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Linear Pursuit-Evasion Games and the Isotropic Rocket

by

Pierre Bernhard

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13. ABSTRACT

~~This work~~ is primarily a study of linear pursuit-evasion games, although several concepts and results are presented that apply to any zero-sum two-person differential game. The direct method of Pontryagin, specifically dealing with linear pursuit-evasion games, is presented and discussed. It is shown how it applies to several information structures. An interesting question is that of the optimality of the strategies generated. It turns out to be closely related to the continuous limit of the discretized information structure used, and of the induced ϵ-strategies. It is shown that the limit strategies are locally optimal. A condition is also found under which there are ϵ-strategies enjoying the same property. The phenomenon that can prevent these strategies from being glo'ally optimal is described, providing criteria to check this optimality. An analysis is given of Pshenichnyi's nonregular points, linking them with the abnormal problem of the calculus of variations and with Isaacs' concept of a barrier. Pontryagin's technique is also applied to multistage games, the main emphasis being on a system-theoretic formulation where the controls are unbounded and the capture set is a subspace. Explicit criteria are given for completion to be possible with the three main information structures. Following Kalman, special attention is given to the case where the capture subspace is a submodule of the system, and his strong controllability theorem is generalized. The second part of this study is an investigation of a specific example; Isaacs' Isotropic Rocket. The previous technique is applied to it, and readily gives interesting results. However, because of the phenomenon mentioned above, the corresponding trajectories, Isaacs' primaries, are not always optimal. The investigation is pursued with the more classical Hamilton-Jacobi theory, generalized by Isaacs to a game-theoretic form of Bellman's

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Wright-Patterson Air Force Base, Ohio

FOREWORD

The work herein was performed under Contract F33615-67-C-1245, Project 5102, Task No. 510215 at the Center for Systems Research, Guidance and Control Laboratory, Stanford University, California.

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This technical report has been reviewed and is approved for publication.

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This work is primarily a study of linear pursuit-evasion games, although several concepts and results are presented that apply to any zero-sum two-person differential game.

The direct method of Pontryagin, specifically dealing with linear pursuit-evasion games, is presented and discussed. It is shown how it applies to several information structures. An interesting question is that of the optimality of the strategies generated. It turns out to be closely related to the continuous limit of the discretized information structure used, and of the induced ϵ-strategies. It is shown that the limit strategies are locally optimal. A condition is also found under which there are ϵ-strategies enjoying the same property. The phenomenon that can prevent these strategies from being globally optimal is described, providing criteria to check this optimality. An analysis is given of Pshenichnyi's nonregular points, linking them with the abnormal problem of the calculus of variations and with Isaacs' concept of a barrier.

Pontryagin's technique is also applied to multistage games, the main emphasis being on a system-theoretic formulation where the controls are unbounded and the capture set is a subspace. Explicit criteria are given for completion to be possible with the three main information structures. Following Kalman, special attention is given to the case where the capture subspace is a submodule of the system, and his strong controllability theorem is generalized.

The second part of this study is an investigation of a specific example; Isaacs' Isotropic Rocket. The previous technique is applied to it, and readily gives interesting results. However, because of the phenomenon mentioned above, the corresponding trajectories, Isaacs' primaries, are not always optimal. The investigation is pursued with the more classical Hamilton-Jacobi theory, generalized by Isaacs to a game-theoretic form of Bellman's dynamic programming.

The game of kind, where the payoff is capture or escape, is first investigated. This determines barriers that can either define a closed capture region or represent discontinuities of the optimal time to go.

The concept of cone of semipermeable directions is emphasized, and a geometrical construction of it is given. This concept is used to present Isaacs' "envelope barrier." It is shown that for a certain range of parameters this barrier does not provide the complete solution of the game of kind. Two other semipermeable surfaces are attached to it, which sometimes succeed in defining a closed capture region. When they do not, two more surfaces are constructed, but they do not close the barrier either. Two new concepts are introduced: the "envelope junction," which is a way in which two semipermeable surfaces can join at a nonzero angle and still form a barrier, and the "singular surface," which is a semipermeable surface, the trajectories of which all come together at a singular point.

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INTRODUCTION

This work is devoted to the study of linear pursuit-evasion games, a special class of two-person zero-sum dynamical games. Dynamical games include differential games, where a continuous system is governed by a set of differential equations, and multistage games, where a discrete system is governed by a set of difference equations.

Problems of pursuit appeared in the Russian literature at an early stage of development of modern control theory. A well-known example is the pursuit problem of Kelendzeridze in the book, Mathematical Theory of Optimal Processes, by Pontryagin, et al. But the full concept of the differential game was first introduced by Rufus Isaacs in various Rand reports as early as 1954, and in his book, Differential Games, in 1965.

The distinctive feature is that both players try to do the best possible with no a priori knowledge of what the opponent is going to do. A striking difference with a two-sided control problem, where one player knows the whole future control of the other one, appears in the fact that while Kelendzeridze was able to apply the maximum principle to the latter case, Pontryagin (Reference [21]) investigated the use of this technique for the former case and found that it usually does not apply.

In the zero-sum game, the only one we shall consider, there is a performance index which one player, whom we shall call P, for pursuer, tries to minimize, and the other player, whom we shall call E, for evader, tries to maximize. Therefore, we have a minimax problem very much like the corresponding case of the classical, static, game theory of Von Neuman and Morgenstern. But here the game has dynamics, and the strategies sought are closed-loop control laws.

A very important question in such games is that of the information structure. At present, the game theoretic form of the most basic concepts and tools to handle partial information--observability, filtering techniques--is missing or unsolved. We shall consider only deterministic structures. It will always be assumed that both players know the state perfectly. The information available to them on each other's control will, however, be varied.

In a pursuit-evasion game, the game is "completed" when the state enters a capture set. The performance index is the time the game lasts until completion, or "capture." The evader tries to survive as long as he can while the pursuer, on the contrary, strives to capture him as quickly as possible. This type of game has provided several of Isaacs' problems. One of these will be considered here, in the second part.

In the first three chapters, we discuss a direct method of Pontryagin, specifically dealing with linear pursuit evasion games.

In the first chapter, this method is introduced and discussed. It is extended slightly to allow various information structures, differing by the amount of information available to the pursuer on the evader's future control, including the case where this whole future is known. The object of this technique is to provide an "estimating function" $T(z)$ such that for a game starting at z, capture is surely possible in a time no longer than $T(z)$.

In the second chapter, our main objective is to study the optimality of the process proposed by Pontryagin. We use, following Fleming and Friedman, ϵ-strategies, consistent with the information structure introduced in the first chapter. We find a condition under which our estimating function can be optimal with such strategies, for small enough ϵ. Then it is seen that, without this condition, $T(z)$ can still be optimum for the continuous process, ϵ-strategies actually yielding capture times arbitrarily close to it as ϵ is decreased. The limit of the ϵ-strategies is investigated and characterized. It is shown, then, how the estimating function and the corresponding strategies can still fail to be optimal, essentially because they can lead to trajectories that would lie inside the capture set for some time before the calculated capture instant. A necessary and sufficient condition for this not to happen is discussed. Finally, non-regular points are briefly investigated, and one kind identified with Isaacs' barriers.

The third chapter is the only one dealing with multistage games. ϵ-strategies naturally lead to a discrete version of the game which is briefly considered, mainly from the point of view of the information

structure. This yields the concepts necessary for applying the previous techniques to the system theoretic formulation of the multistage game, with unbounded controls. In that case, the estimating function is shown to be optimal. The particular case where the capture subspace is a submodule is investigated and an earlier result by Kalman [19] generalized.

The remaining three chapters deal with a specific example: Isaacs' Isotropic Rocket Game, described in [17] and [18].

In the fourth chapter, this game is introduced, and its various descriptions presented. Then we apply to it the method of the first chapter, which rapidly yields an estimating function. However, this estimating function is not optimal in the whole state space, the condition of Chapter Two being violated.

In the fifth chapter, we use Isaacs' technique and try to solve the "game of kind" qualitative game, the outcome of which is capture or escape. It turns out that Isaacs' conjectures, according to which his solution would have been complete, are not verified for all values of some parameters. Trying to complete this solution leads to the concept of envelope junction, a corner condition for barriers. A new type of barrier is also introduced, the singular barrier, where all member trajectories meet at a singular point. But we have not been able to finish the problem completely, due to the fact that the solution seems to be linked to another unsolved problem, of the following chapter.

The sixth and last chapter deals with the "game of degree" quantitative game, the outcome of which is longer or shorter capture time. It is shown that the solution involves a "safe contact," first perceived by J. V. Breakwell and already investigated under his supervision. The occurrence of this safe contact is linked to the phenomenon pointed out in Chapter Two, causing the estimating function to be non-optimal. Then, we need to allow a corner in the optimal trajectories. The general corner condition for differential games with integral payoff is derived, largely using the concept of field, and resting upon ideas developed by Isaacs and Breakwell. However, the partial differential equation it leads to in this game is so complicated that we were unable to integrate it numeri-

cally. Some analytical results are derived concerning the qualitative shape of its solution, and conjectures presented on how the solution of the game may look.

1. THE DIRECT METHOD OF PONTRYAGIN

In this chapter, we present the method developed by Pontryagin [22, 23,24] to deal with linear pursuit-evasion games. In doing so, we discuss how the method can apply to various information structures.

1.1 Statement of the Problem

In an n-dimensional Euclidean vector space E, a system is governed by the differential equation

$$\frac{dz}{dt} = C\,z - u + v \tag{1.1}$$

where

z is the state of the system $z \in E$;

C is a constant $n \times n$ matrix;

$u \in P$ is a control variable chosen by the pursuer;

$v \in Q$ is a control variable chosen by the evader;

P and Q are closed convex subsets of E.

Admissible control functions $u(\cdot)$ and $v(\cdot)$ are measurable functions of time, taking their values in P and Q respectively.

A given subspace L of E is called the geometrical space. The orthogonal projection of E onto L is called π.

A given closed convex subset $\mathcal{C}$ of L is called the capture set. The dimension of $\mathcal{C}$ can be n. In that case, $L = E$, and the operator π is simply the identity.

Capture is defined as

$$\pi\, z \in \mathcal{C}\,.$$

The general problem can be stated as deciding whether it is possible, knowing the state z at each instant, and with some suitable information on the evader's control, to construct a control function $u(\cdot)$ such that capture will eventually occur. If this is possible, in what time, and how should the control u be chosen?

The question of the evader's "best" behavior will be considered later.

1.2 Remarks

Before going into the analysis of this problem, we shall make some remarks about its formulation.

i) Dynamics. The dynamics described in (1.1) may seem somewhat restrictive as compared to the more general formulation

$$\dot{z} = C\,z - Gu' + Jv' \;. \tag{1.1a}$$

However, we restrict u' and v' to belong to compact convex subsets. Therefore, (1.1a) is equivalent to (1.1) by letting

$$Gu' = u \qquad\qquad Jv' = v$$

and it is straightforward to see that if u' and v' belong to P' and Q', closed and convex in their respective spaces, u and v belong to closed convex subsets P and Q of E.

The use of constant sets P and Q corresponds to constant matrices G and J, which is consistent with the fact that we take a constant matrix C.

ii) Many-Control System. Another generalization one might want to consider is a many-control system:

$$\dot{z} = C\,z + u_1 + u_2 + \dots + u_p$$

$$u_i \in P_i \;.$$

But the approach we take is essentially unsymmetric. We investigate what can be done with the control u, knowing how the other control can act on the system. Whether v is under the control of a single player or is the added effect of several players' controls makes no difference.

We can reduce this formulation to the first one, letting

$$-u_1 = u \in P = -P_1$$

$$u_2 + \dots + u_p = v \in Q = P_2 + \dots + P_q \;.$$

And notice that Q is still convex.

iii) Convexity. We insist on the compactness and convexity of the sets P and Q because the following theory, at several points, depends critically on it. It is, therefore, interesting to notice that, in assuming the convexity of these sets, there is no loss of generality. It is a well-known fact in control theory that, for a dynamical system, any point of the convex hull of a non-convex control set could be approximated as closely as desired by chattering between n values belonging to the control set itself.

The convexity of the capture set e is needed as well, and must be regarded as a restrictive assumption. Notice that it is verified by the interesting special case

$$\mathcal{C} = \{0\}$$

which corresponds to capture being defined as

$$z \in M \qquad L \oplus M = E \qquad M \text{ orthogonal complement to } L$$

1.3 The Information Structure

Following Pontryagin, we introduce a special information structure, which we shall call the lower rule ϵ.

At each instant t the pursuer knows the state $z(t)$ and the evader's control history $v(t)$ for a time ϵ in the future, namely

$$v(s) \qquad t \leq s \leq t + \epsilon \qquad \text{written} \qquad v[t,t+\epsilon] \; .$$

A pursuit strategy, then, is a mapping

$$\bar{u} : [t_o,t_o+\epsilon] \times E \times Q^{[t_o,t_o+\epsilon]} \to P$$

written as

$$u = \bar{u}\Big(t;z(t_o),v[t_o,t_o+\epsilon]\Big) \qquad t_o \leq t \leq t_o + \epsilon \; .$$

In this chapter, this definition will be enough for our purposes.

As the system is time-invariant, we can arbitrarily set $t_o = 0$.

The question of the updating of the function $\bar{u}$ as time goes on will be discussed in the second chapter (Section 2.3).

Three cases will be considered:

α) ϵ as big as desired by the pursuer. The evader's control is known by the pursuer for the whole future.

This reduces the problem to a classical formulation, used by Kelendzeridze, Varaiya [27], Kalman [19], and other authors. Does there exist, for every control history $v(\cdot)$ a corresponding control history $u(\cdot)$ such that capture occurs in finite time?

β) ϵ is a given positive number, possibly very small. This is the case considered by Pontryagin. He gives, in addition, a time of capture valid for every positive ϵ . This suggests the third case:

γ) $\epsilon = 0$: Only the present value of the evader's control is known. This is again a classical problem, considered by Pontryagin himself in [21,24], and other authors. This knowledge has been found, in some cases, to be necessary for optimal strategies to exist.

We shall consider this case as a limit of the previous one, following Fleming [11,12,13] and Friedman [14,15].

1.4 A Sufficient Condition

We want to find a sufficient condition for capture to be possible, and an estimating function

$$T : E \rightarrow R$$

such that a state z can surely be captured in a time no longer than $T(z)$. Pontryagin reaches this objective by constructing a set of capturable points V_τ , a function of the real variable τ , such that

- $\pi V_o = C$

- if $z_o \in V_{\tau_o}$, there exists a pursuit control such that, with $z(0) = z_o$, the solution of (1.1) verifies $z(\epsilon) \in V_{\tau_o - \epsilon}$.

As a consequence of the second property, at time ϵ there exists a new pursuit control insuring the same inclusion at time 2ϵ and so on. Thus, eventually

$$z(\tau_o) \in V_o \qquad \text{or} \qquad \pi\, z(\tau_o) \in \mathcal{C}\,.$$

Therefore, any mapping $T(\cdot)$ of the state space into the reals such that

$$z \in V_{T(z)}$$

is an estimating function.

From now on, the estimating function we consider shall always be the smallest real number $T(z)$ such that the above inclusion is verified.

The construction of V_τ can depend on ϵ. We must clearly have

$$V_\tau(\epsilon_1) \subseteq V_\tau(\epsilon_2) \qquad \forall\, \epsilon_1 \leq \epsilon_2\,,\ \forall\, \tau$$

since the control $\bar{u}$ constructed with the rule ϵ_1 can also be constructed with the rule ϵ_2. We shall also see a construction, proposed by Pontryagin, giving a set V_τ valid for every positive ϵ.

We shall actually find a family of sets W_τ in L, and define V_τ as

$$V_\tau = \Phi(-\tau)\pi^{-1} W_\tau = \{z \,|\, \pi\Phi(\tau) z \in W_\tau\}$$

where $\Phi(\tau) = e^{\tau C}$ is the transition matrix associated with C. We shall have

$$W_o = \mathcal{C} \qquad\qquad V_o = \{z \,|\, \pi\, z \in \mathcal{C}\}$$

so that the family V_τ verifies the first condition $\pi\, V_o = \mathcal{C}$.

1.5 Properties of the Sets W_τ

It is straightforward to verify that the solutions of (1.1) verify

$$\pi\Phi(\tau_o-t)z(t) = \pi\Phi(\tau_o)z(0) + \pi\int_{\tau_o-t}^{\tau_o}\Phi(r)[v(\tau_o-r) - u(\tau_o-r)]dr \; . \tag{1.2}$$

To simplify the equations, we introduce the notations

$$u_r = \pi\Phi(r)u \qquad P_r = \pi\Phi(r)P$$

$$v_r = \pi\Phi(r)v \qquad Q_r = \pi\Phi(r)Q \; .$$

Now, assume that we have the inclusion

$$\pi\Phi(\tau_o)z(0) \in W_{\tau_o} \; . \tag{1.3}$$

Given $v[0,\epsilon]$ we want to be able to construct a control $u[0,\epsilon]$ such that

$$\pi\Phi(\tau_o)z(0) + \int_{\tau_o-\epsilon}^{\tau_o} v_r(\tau-r)dr - \int_{\tau_o-\epsilon}^{\tau_o} u_r(\tau-r)dr \in W_{\tau-\epsilon} \; .$$

Or, defining the sum of a set and a vector in the usual way

$$\pi\Phi(\tau_o)z(0) + \int_{\tau_o-\epsilon}^{\tau_o} v_r(\tau-r)dr \in W_{\tau-\epsilon} + \int_{\tau_o-\epsilon}^{\tau_o} u_r(\tau-r)dr \; .$$

Define the integral of a set P_r as the set of all possible integrals of functions u_r taking their values in P_r. The existence of a function u_r verifying the last inclusion is equivalent to

$$\pi\Phi(\tau_o)z(0) + \int_{\tau_o-\epsilon}^{\tau_o} v_r(\tau-r)dr \in W_{\tau-\epsilon} + \int_{\tau_o-\epsilon}^{\tau_o} P_r dr \; .$$

This must be true for every possible control $v(\cdot)$, so this is equivalent to

$$\pi\Phi(\tau_o)z(0) + \int_{\tau_o-\epsilon}^{\tau_o} Q_r dr \subseteq W_{\tau_o-\epsilon} + \int_{\tau_o-\epsilon}^{\tau_o} P_r dr \;. \tag{1.3a}$$

Finally, as we want this to be true for every point verifying (1.3), this, in turn, is equivalent to

$$W_{\tau_o} + \int_{\tau_o-\epsilon}^{\tau_o} Q_r dr \subseteq W_{\tau_o-\epsilon} + \int_{\tau_o-\epsilon}^{\tau_o} P_r dr \;, \tag{1.4}$$

which is the characteristic property of the sets W_τ .

1.6 Geometric Subtraction

In order to ease the handling of relation (1.4), Pontryagin introduces the following operation:

Given two subsets A and B of a vectorial space, define their <u>geometric difference</u> as

$$D = A \overset{*}{-} B \qquad\qquad D = \{z \mid B + z \subseteq A\}$$

which means that D is the biggest set such that

$$D + B \subseteq A \;.$$

If D is non-empty, we say that <u>A star includes B</u> : $A \overset{*}{\supset} B$;
we say that the property of <u>complete sweeping</u> is verified if: $A = B + D$.

<u>Proposition:</u>

i) if A and B are convex, their geometric difference is convex;

ii) $A \overset{*}{-} (B+C) = (A \overset{*}{-} B) \overset{*}{-} C = A \overset{*}{-} B \overset{*}{-} C = A \overset{*}{-} C \overset{*}{-} B$;

iii) $(A+B) \overset{*}{-} C \supseteq (A \overset{*}{-} C) + B$.

<u>Proof:</u>

i) Let $D = A \overset{*}{-} B$ be non-empty. Let d_1 and d_2 belong to D . By definition, for every $b \in B$, their exists an $a_i(b) \in A$ such that

$$d_i + b + a_i(b) \;, \quad i = 1,2 \;.$$

Consider $d = \alpha d_1 + (1-\alpha)d_2$ and check that $a(b) = \alpha a_1(b) + (1-\alpha)a_2(b)$ verifies

$$d + b = \alpha(d_1+b) + (1-\alpha)(d_2+b) = a(b) \quad \forall b$$

if A is convex, $a(b) \in A$ and the first result is proved. Notice that we do not need the convexity of B. In the sequel, however, only convex sets will be met. There is consequently no point in stating the result with more generality.

ii) Consider $A = (A \overset{*}{-} B) \overset{*}{-} C$, and $d \in D$, we have, by definition,

$$d \in (A \overset{*}{-} B) \overset{*}{-} C \iff d + C \subseteq (A \overset{*}{-} B) \iff (d+C) + B \subseteq A .$$

Now, the addition $(d+C) + B$ is associative; thus, the last relation is equivalent to

$$d + (C+B) \subseteq A \iff d \in A \overset{*}{-} (C+B) ,$$

which proves the second result.

iii) Consider $D = (A \overset{*}{-} C) + B$, and $d \in D$. By definition, there exists $e \in A \overset{*}{-} C$ and $b \in B$ such that $d = e + b$. Now we have

$$e + C \subset A \Rightarrow d + C \subseteq A + b \subseteq A + B \Rightarrow d \in (A+B) \overset{*}{-} C$$

which proves the third result. Notice that it is generally not true that $(A+B) \overset{*}{-} C = (A \overset{*}{-} C) + B$. It is enough, to see it, to take $B = C$ and a case where $A \overset{*}{-} C$ does not have complete sweeping (see Fig. 1). However, we have the following simple but interesting property:

Proposition:

If $A \overset{*}{-} C$ has the property of complete sweeping, then this is true for $(A+B) \overset{*}{-} C$ and $(A+B) \overset{*}{-} C = (A \overset{*}{-} C) + B$.

Proof:

Let $D = (A \overset{*}{-} C) + B$.

Then $D + C = (A \overset{*}{-} C) + C + B = A + B$,

the last equality because of the assumption of complete sweeping on $A \overset{*}{-} C$. And this proves the claim.

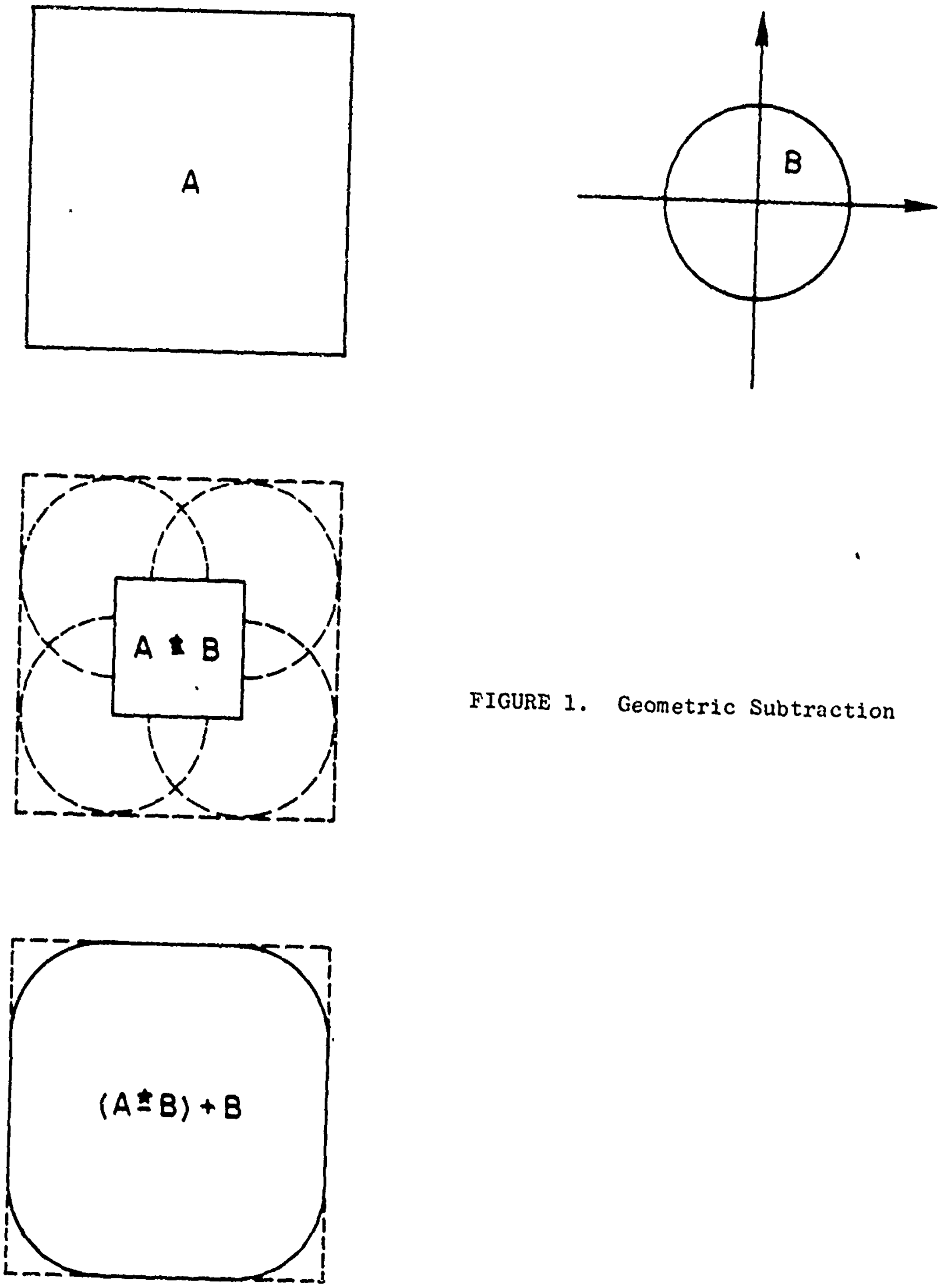

FIGURE 1. Geometric Subtraction

1.7 Construction of the Sets W_τ

Relation (1.4) can be written as

$$W_\tau \subseteq W_{\tau-\epsilon} + \int_{\tau-\epsilon}^{\tau} P_r dr \ \overset{*}{-} \int_{\tau-\epsilon}^{\tau} Q_r dr \ . \tag{1.5}$$

Depending on the information structure, we can use (1.5) in different ways. We consider the three cases of Section (1.3):

α) ϵ arbitrarily large. We can use $\epsilon = \tau$, and find immediately the set we call $W_\tau^{(\infty)}$

$$W_\tau^{(\infty)} = \left(\mathcal{C} + \int_0^\tau P_r dr\right) \overset{*}{-} \int_0^\tau Q_r dr \ .$$

However, it is easy to see that this family will generally not verify (1.4) for a smaller ϵ. In fact, we have, using the first proposition of Section 1.6,

$$W_\tau^{(\infty)} = \left(\mathcal{C} + \int_0^{\tau-\epsilon} P_r dr + \int_{\tau-\epsilon}^{\tau} P_r dr\right) \overset{*}{-} \int_0^{\tau-\epsilon} Q_r dr \overset{*}{-} \int_{\tau-\epsilon}^{\tau} Q_r dr$$

$$\supseteq \left[\left(\mathcal{C} + \int_0^{\tau-\epsilon} P_r dr\right) \overset{*}{-} \int_0^{\tau-\epsilon} Q_r dr + \int_{\tau-\epsilon}^{\tau} P_r dr\right] \overset{*}{-} \int_{\tau-\epsilon}^{\tau} Q_r dr$$

where the right hand side is the same as in (1.5). However, because in general the addition and geometric subtraction of $\int_{\tau-\epsilon}^{\tau} P_r dr$ and $\int_0^{\tau-\epsilon} Q_r dr$ do not commute, the inclusion is strict and (1.4) violated.

This set is nevertheless very important, because only when it does verify (1.4) is an ϵ-strategy optimal. We shall discuss this point in Chapter 2.

β) ϵ given positive. We use the induction argument of Section 1.4, and construct the corresponding set $W_\tau^{(\epsilon)}$

$$W_\tau^{(\epsilon)} = \left(\left(---- \left(\left(\left(\left(\mathcal{C} + \int_0^\epsilon P_r dr\right) \overset{*}{-} \int_0^\epsilon Q_r dr\right) + \int_\epsilon^{2\epsilon} P_r dr\right)\right.\right.\right.$$

$$\left.\left.\left. \overset{*}{-} \int_\epsilon^{2\epsilon} Q_r dr\right) ------\right) + \int_{(n-1)\epsilon}^{n\epsilon} P_r dr\right) \overset{*}{-} \int_{(n-1)\epsilon}^{n\epsilon} Q_r dr \; ,$$

$$n\epsilon = \tau \; .$$

This construction is called by Pontryagin <u>the alternating sum</u> of the sets

$$U_k = \int_{(k-1)\epsilon}^{k\epsilon} P_r dr \quad \text{and} \quad V_k \int_{(k-1)\epsilon}^{k\epsilon} Q_r dr \; .$$

Our set $W_\tau^{(\epsilon)}$ is defined only for discrete values of τ. $T(z)$, thus, takes only discrete values, but this is not in contradiction with our theory so far.

For the continuous case, we will use another, continuous, definition, but this one will be used for the case of discrete systems in Chapter 3.

γ) <u>ϵ goes to zero</u>. We define a set $W_\tau^{(o)}$ as a limit of the previous construction. The precise topology involved is discussed in [23] and [24]; it will be presented in Section 2.6. For the time being, we shall consider it as a pointwise limit: every point belongs to it which can be approximated as closely as desired by a point of a $W_\tau^{(\epsilon)}$. In other words,

$$W_\tau^{(o)} = \left\{ z \,\middle|\, \exists\, z_{\delta_i} \in W_\tau^{(\delta_i)} : \|z - z_{\delta_i}\| \to 0 \quad \text{as} \quad \delta_i \to 0 \right\} .$$

It clearly gives the compact set. This set is called by Pontryagin the alternating integral and denoted by

$$W_\tau^{(o)} = \int_{\mathcal{C},0}^{\tau} \left[P_r \overset{*}{-} Q_r\right] dr \; .$$

Notice that this is a mere notation. $P_r \overset{*}{-} Q_r$ may not exist, and $W_\tau^{(o)}$ still be non-empty. Again because of the properties

of the geometric difference, the set $W_\tau^{(\delta)}$ verifies (1.4) as soon as $\delta \leq \epsilon$. Thus $W_\tau^{(o)}$ verifies it for every ϵ .

δ) $\epsilon = 0$: another definition. Finally, in the case where $P_r \,\bar{\supset}\, Q_r$ for every r , we can consider another set family

$$W'_\tau = \mathcal{C} + \int_o^\tau (P_r \overset{*}{-} Q_r)dr$$

and verify directly that it has the required property (1.4):

$$W'_\tau = \left(\mathcal{C} + \int_o^{\tau-\epsilon} (P_r \overset{*}{-} Q_r)dr\right) + \int_{\tau-\epsilon}^{\tau} (P_r \overset{*}{-} Q_r)dr$$

$$\subseteq W'_{\tau-\epsilon} + \left(\int_{\tau-\epsilon}^{\tau} P_r dr \overset{*}{-} \int_{\tau-\epsilon}^{\tau} Q_r dr\right)$$

$$\subseteq \left(W'_{\tau-\epsilon} + \int_{\tau-\epsilon}^{\tau} P_r dr\right) \overset{*}{-} \int_{\tau-\epsilon}^{\tau} Q_r dr .$$

Notice the particular feature of this construction: the existence of the corresponding family of sets W'_τ depends only on the existence of $P_r \overset{*}{-} Q_r$. Therefore, if for some capture set $\mathcal{C}$ this family exists, this will be true with every capture set, including $\mathcal{C} = \{0\}$, or capture defined as point coincidence.

Besides the point we just mentioned, the use of this definition will be interesting in the following discussion.

1.8 Relative Size of the W'_τs

Let us first compare $W_\tau^{(\epsilon_1)}$ to $W_\tau^{(\epsilon_2)}$ with $\epsilon_2 > \epsilon_1$

$$W_{\epsilon_2}^{(\epsilon_2)} = \left(\mathcal{C} + \int_o^{\epsilon_2} P_r dr\right) \overset{*}{-} \int_o^{\epsilon_2} Q_r dr$$

$$\supseteq \left[\left(\mathcal{C} + \int_o^{\epsilon_1} P_r dr\right) \overset{*}{-} \int_o^{\epsilon_1} Q_r dr + \int_{\epsilon_1}^{\epsilon_2} P_r dr\right] \overset{*}{-} \int_{\epsilon_1}^{\epsilon_2} Q_r dr$$

$$W_{\epsilon_2}^{(\epsilon_2)} \supseteq W_{\epsilon_1}^{(\epsilon_1)} + \int_{\epsilon_1}^{\epsilon_2} P_r dr \stackrel{*}{-} \int_{\epsilon_1}^{\epsilon_2} Q_r dr .$$

If $\epsilon_2 = 2\epsilon_1$, the last expression is $W_{\epsilon_2}^{(\epsilon_1)}$. This argument used recursively shows that, when they are both defined,

$$W_\tau^{(\epsilon_2)} \supseteq W_\tau^{(\epsilon_1)} ,$$

and thus, in the limit as ϵ goes to zero

$$W_\tau^{(\epsilon)} \supseteq W_\tau^{(o)} .$$

Using the same calculation to compare $W_\tau^{(\infty)}$ to $W_\tau^{(\epsilon)}$ where $\epsilon = \tau/n$, it is seen that

$$W_\tau^{(\infty)} \supseteq W_\tau^{(\epsilon)} .$$

Finally, we compare W'_τ to any $W_\tau^{(\epsilon)}$. Using recursively the following calculation:

$$W'_\tau = W'_{\tau-\epsilon} + \int_{\tau-\epsilon}^{\tau} (P_r \stackrel{*}{-} Q_r) dr \subseteq W'_{\tau-\epsilon} + \left(\int_{\tau-\epsilon}^{\tau} P_r dr \stackrel{*}{-} \int_{\tau-\epsilon}^{\tau} Q_r dr \right)$$

$$\subseteq \left(W'_{\tau-\epsilon} + \int_{\tau-\epsilon}^{\tau} P_r dr \right) \stackrel{*}{-} \int_{\tau-\epsilon}^{\tau} Q_r dr$$

for $\tau = \epsilon$, then 2ϵ , and so on, shows that

$$W'_\tau \subseteq W_\tau^{(\epsilon)} \qquad \forall\ \epsilon = \frac{\tau}{n} \qquad n \in \mathbb{N} ,$$

and thus, taking into account the other three inclusions derived:

$$W'_\tau \subseteq W_\tau^{(o)} \subseteq W_\tau^{(\epsilon_1)} \subseteq W_\tau^{(\epsilon_2)} \subseteq W_\tau^{(\infty)} .$$

If, now, we remember the definition proposed for $T(z)$, each of these sets gives a corresponding estimating function, and the above inclusions translate into

$$T'(z) \geq T^{(o)}(z) \geq T^{(\epsilon_1)}(z) \geq T^{(\epsilon_2)}(z) \geq T^{(\infty)}(z)$$

where

$$T(z) = \infty \quad \text{if} \quad \pi\Phi(\tau)z \not\subset W_\tau \quad \forall\tau \ .$$

The last three inequalities simply say that the time we know to be sufficient to capture increases as the amount of information available on the evader's control decreases. This agrees with what we said in Section 1.4.

An interesting simplification occurs when $P_r \supseteq Q_r$ for every r, with complete sweeping. Then, W'_τ exists, and we have

$$W'_\tau + \int_0^\tau Q_r dr = \mathcal{C} + \int_0^\tau \left[(P_r \overset{*}{-} Q_r) + Q_r\right] dr = \mathcal{C} + \int_0^\tau P_r dr$$

and thus

$$W'_\tau = \left(\mathcal{C} + \int_0^\tau P_r dr\right) \overset{*}{-} \int_0^\tau Q_r dr = W_\tau^{(\infty)}$$

which, in view of the chain of inclusions derived, proves that the four constructions give the same set W_τ, and we can use the most convenient construction, W'_τ, for instance.

In consequence also, the function $T(z)$ does not depend on the information structure. Assuming more knowledge of the evader's strategy does not allow us to improve our a priori estimate of the capture time. However, once $v(\cdot)$ is actually known, we might be able to take better advantage of it and capture in a shorter time.

2. OPTIMALITY

In this chapter, we address ourselves to two problems which are to a great extent interconnected: the question of the optimality of the process we describe, and the question of its limit as ϵ goes to zero, what we shall call the limit process. We shall also be obliged to introduce the motion of regular point, and we shall make a brief analysis of non-regular points.

2.1 The Concepts of Optimality

The approach taken so far is essentially unsymmetric. We have assumed that the pursuer has an information advantage over the evader. This allows us to ask for a strong type of optimality: we want to use optimally the information available. We seek a function

$$u^o\Big(t; z(t_o), v[t_o, t_o+\epsilon]\Big)$$

providing the minimum capture time over all such functions for every admissible history $v[t_o, t_o+\epsilon]$. Let $J(u,v)$ be the actual capture time

$$J(u^o, v) = \min_u J(u,v) \ .$$

Now, if the evader plays optimally, he will choose his control in such a way as to maximize the above functional, so that

$$J(u^o, v^o) = \max_v \min_u J(u,v) \ .$$

In the classical saddle point formulation, one only seeks a control optimal against the opponent's saddle point control. Here, we want to know how to modify the pursuer's control to take into advantage a possible deviation of the evader from his optimal control, and we do not require that there exist a saddle point, although we shall see further in what sense there is one.

It is interesting at this point to review the exact relation between the maximin operation and the saddle point.

Let $J(u,v)$ be a continuous functional:

$$J : P \times Q \to \mathbb{R}$$

where P and Q are compact subsets of topological spaces A and B. Then J reaches its extrema on $P \times Q$.

Let

$$\operatorname{Arg} \min_{u \in P} J(u,v) = u^{o}(v) \ , \ \operatorname{Arg} \max_{v \in Q} J(u,v) = v^{o}(u) \ .$$

Proposition: The existence of a saddle point:

$$J(u^*,v) \leq J(u^*,v^*) \leq J(u,v^*) \qquad \forall (u,v) \in P \times Q \tag{2.1}$$

is equivalent to

$$\min_{u \in P} \max_{v \in Q} J(u,v) = \max_{v \in Q} \min_{u \in P} J(u,v) = J(u^*,v^*)$$

(2.1a)

$$u^{o}(v^*) = u^* \qquad\qquad v^{o}(u^*) = v^*$$

Proof: (2.1a) implies (2.1) trivially, by the definition of u^{o} and v^{o}. Let us prove that (2.1) implies (2.1a):

by definition of v^{o},

$$J\left(u^*,v^{o}(u^*)\right) \geq J(u^*,v^*)$$

and by (2.1)

$$J\left(u^*,v^{o}(u^*)\right) \leq J(u^*,v^*) \ .$$

Thus

$$J\left(u^*,v^{o}(u^*)\right) = J(u^*,v^*) \qquad v^{o}(u^*) = v^* \tag{2.2}$$

where this definition of v^{o} is valid, if not necessarily unique. Now, let u_o provide the minimax:

$$J\left(u_o,v^{o}(u_o)\right) \leq J\left(u,v^{o}(u)\right) \qquad \forall \ u \in P \ .$$

Apply this with $u = u^*$, and use (2.2):

$$J(u^*,v^*) = J\left(u^*,v^{o}(u^*)\right) \geq J\left(u_o,v^{o}(u_o)\right) .$$

By definition of v^o

$$J\left(u_o, v^o(u_o)\right) \geq J(u_o, v^*)$$

by (2.1)

$$J(u_o, v^*) \geq J(u^*, v^*) \ .$$

Therefore, all three inequalities reduce to equalities, proving that

$$\min_u \max_v J(u,v) = J(u^*,v^*) \qquad v^o(u^*) = v^* \ .$$

Clearly, the proof can be repeated <u>mutis mutandis</u> to derive the rest of (2.1a), proving the proposition. If the mapping J is injective (one to one), then we also have $u_o = u^*$, $v_c = v^*$ uniquely.

The technique we shall use to construct the control u will be to have the estimating function decrease as fast as possible. However, this does not guarantee optimality with what we know of the estimating function. Actually, defined as a sufficient time for capture, it is not even necessarily unique. We have no proof that another $\hat{T}(z)$ could not be found, that would lead to a different strategy.

But assume now that at each instant, E can insure that capture will not occur in a time less than $T(z)$. Then, having this function decrease as fast as possible will indeed be the optimal behavior for P . $T(z)$ will then be said to be optimal.

Therefore, we are led to the investigation of the maximin strategies, and of the optimality of the sufficient capture time we have displayed.

2.2 A Result by Gusyatnikov and Nikolsky

In [16], Gusyatnikov and Nikolsky give a sufficient condition for $T(z)$ as defined here to be optimal, by displaying a δ-efficient strategy for any positive δ , for the evader. This condition is the following:

<u>Condition A</u>:

- $P_\tau \supsetneq Q_\tau \qquad \forall\tau$, complete sweeping,
- $\forall\, u \in P \ , \ \exists\, v \in Q : \ u_\tau - v_\tau \in P_\tau \overset{*}{-} Q_\tau \qquad \forall\tau \ .$ (2.3)

Condition A says that the element v which provides the complement v_τ to u_τ in $P_\tau \overset{*}{-} Q_\tau$ is independent of τ. In particular, if for some τ_o, v is unique, then this v will verify (2.3) for every τ.

Under this condition, a δ-efficient strategy is displayed, of the form

$$v^{o\prime} = v\Big(t, z(t_o) \;,\; u[t_o-\epsilon, t_o]\Big)$$

in terms of the constant complement of u, And for every positive δ, there is an ϵ small enough so that this strategy insures that capture will not occur in a time less than $T\big(z(0)\big) - \delta$. (This is the definition of a δ-efficient strategy.)

We see then that it should be possible to define properly the limit of this strategy, which would yield the capture time $T\big(z(0)\big)$. Actually, we shall try to display an optimal strategy (0-efficient) with a finite ϵ, the limit of which as ϵ goes to zero will be obvious.

2.3 Jumps of the Estimating Function

To analyze the variation of the estimating function, we introduce the concept of jumps of the estimating function (see [22] and [24]). We know that if

$$z_o \in W_{\tau_o} \;,$$

then, because relation (1.3a) is verified, there exists a control $u(\cdot)$ such that

$$z(\epsilon) \in W_{\tau_o - \epsilon} \qquad \text{if} \qquad z(0) = z_o \;.$$

However, relation (1.3a) might be verified for some smaller τ_1. If τ_1 is the smallest such τ for which it holds, we can find a control $u^o(\cdot)$ such that

$$z(\epsilon) \in W_{\tau_1 - \epsilon} \qquad\qquad T\big(z(\epsilon)\big) = \tau_1 - \epsilon \le \tau_o - \epsilon \;.$$

Let

$$\delta_\tau = \tau_o - \tau_1 .$$

We shall say that with the strategy $v(\cdot)$, the estimating function has a jump $\delta\tau$ at z_o. The behavior we propose for the evader is to make this jump as small as possible, and, if possible, zero.

At this point, a difficulty arises. At time τ_o, the evader chooses a control $v^o[t_o, t_o+\epsilon]$. Assume he can choose it such that the corresponding $\delta\tau$ be null. Let t_1 belong to the interval $(t_o, t_o+\epsilon)$. From $z(t_1)$, under the previous controls, assume there exists a control $v'[t_1, t_1+\epsilon]$ that avoids a jump in the estimating function as well. We are not assured that v^o and v' agree on $[t_1, t_o+\epsilon]$, so that although such a control v' exists, the evader might not be able to use it and might let a non-zero jump occur.

To solve, or rather eliminate, this problem, we use Fleming's definition of a strategy (introduced in [11]). We assume that both players choose their controls at time t_o for the whole interval $(t_o, t_o+\epsilon)$ and play them. Consequently, the question of updating the control function at an interior point becomes irrelevant. This is Fleming's minorant game or Friedman's lower ϵ-strategy (hence the terminology "lower rule ϵ"). A continuous strategy will be, by definition, the limit of such a process.

It was proved by Fleming [12,13] that the value of this discretized game has a limit as ϵ goes to zero. Moreover, this limit is the same as the limit of the majorant game, defined in the same way but giving the evader the information advantage (Friedman's upper ϵ-strategy). Fleming's proofs are made discretizing also the dynamics, and allowing, then, for mixed strategies at each move (and for fixed duration games). Friedman [14,15], generalizing a work of Varaiya and Lin, gave the proof using continuous dynamics, and, in [15], for a class of games including ours, with pure strategies.

Therefore, we know our game has a saddle point, and that the limit of the maximin capture time is the saddle point value of the game. From now on, we shall investigate this lower ϵ-strategy. We shall see under

what conditions the evader can always prevent a jump. We shall take $t_o = 0$ since the game is time-invariant.

2.4 A First Necessary Condition

We are seeking an optimal ϵ-strategy. If a strategy $v[0,\epsilon]$ is such that, for $\pi\Phi(\tau_o)z_o \in W_{\tau_o}$,

$$\pi\Phi(\tau_o)z_o + \pi\int_{\tau_o-\epsilon}^{\tau_o} \Phi(r)v(\tau_o-r)dr \in \operatorname{int}\left\{W_{\tau_o-\epsilon} + \int_{\tau_o-\epsilon}^{\tau_o} P_r dr\right\},$$

then, by continuity, this is true for some τ smaller than τ_o, and there will be a jump in $T(z)$. Thus, a necessary condition on v is that

$$\pi\Phi(\tau_o)z_o + \pi\int_{\tau_o-\epsilon}^{\tau_o} \Phi(r)v(\tau_o-r)dr \in \partial\left\{W_{\tau_o-\epsilon} + \int_{\tau_o-\epsilon}^{\tau_o} P_r dr\right\}. \tag{2.4}$$

A necessary condition for this to be always possible is

$$W_{\tau_o} + \pi\int_{\tau_o-\epsilon}^{\tau_o} \Phi(r)Qdr = W_{\tau_o-\epsilon} + \pi\int_{\tau_o-\epsilon}^{\tau_o} \Phi(r)Pdr. \tag{2.5}$$

Then, because $\pi\Phi(\tau_o)z_o \in \partial W_{\tau_o}$, W_{τ_o} is convex and $\int Q_r dr$ is compact, there always exists a control v^o such that

$$\pi\Phi(\tau_o)z_o + \pi\int_{\tau_o-\epsilon}^{\tau_o} Q(r)v^o(\tau_o-r)dr \in \partial\left\{W_{\tau_o} + \pi\int_{\tau_o-\epsilon}^{\tau_o} \Phi(r)Qdr\right\}$$

$$= \partial\left\{W_{\tau_o-\epsilon} + \pi\int_{\tau_o-\epsilon}^{\tau_o} \Phi(r)Pdr\right\}.$$

Proposition: Condition (2.5) implies

$$W_\tau = \mathcal{C} + \int_0^\tau P_r dr \pm \int_0^\tau Q_r dr \text{ , complete sweeping.}$$

Proof: We first prove that the fact that it is true for every sufficiently small ϵ, for every τ, implies that it is true for every ϵ:

$$W_\tau + \int_{\tau-2\epsilon}^{\tau} Q_r dr = \left(W_\tau + \int_{\tau-\epsilon}^{\tau} Q_r dr\right) + \int_{\tau-2\epsilon}^{\tau-\epsilon} Q_r dr$$

$$= W_{\tau-\epsilon} + \int_{\tau-2\epsilon}^{\tau-\epsilon} Q_r dr + \int_{\tau-\epsilon}^{\tau} P_r dr$$

$$= W_{\tau-2\epsilon} + \int_{\tau-2\epsilon}^{\tau-\epsilon} P_r dr + \int_{\tau-\epsilon}^{\tau} P_r dr \text{ ,}$$

and this proves this first claim. Then, taking $\epsilon = \tau$, it becomes

$$W_\tau + \int_0^\tau Q_r dr = \mathcal{C} + \int_0^\tau P_r dr \text{ ,}$$

which proves the proposition.

In this case, thus, $W_\tau^{(o)} = W_\tau^{(\infty)}$. But W'_τ may be smaller or not exist, since we do not require that $P_r \ni Q_r$, and if it does, it may not have complete sweeping.

2.5 Characterization of the Strategies

From the previous remarks, we can characterize the candidate optimal strategies v^o and u^o. We need the following simple result:

Lemma: Let W be a closed convex set in a Euclidean vector space. Let Λ be a closed set. Let $\zeta \in \partial W$, and let n be a normal to W at ζ (normal in the definition related to convex sets; see [26]). Finally, let $\lambda \in \Lambda$ be such that

$$\langle n,\lambda\rangle = \max_{\mu \in \Lambda} \langle n,\mu\rangle \text{ .}$$

Then $\lambda \in \partial\Lambda$ and $\xi = \zeta + \lambda \in \partial(W+\Lambda)$. Conversely, if $\xi \in \partial(W+\Lambda)$, there is a normal n to W at ζ such that $\langle n,\lambda\rangle$ is maximized.

Proof: The first part of the claim is trivial. Let Π be the tangent hyperplane normal to n at ζ. Assume ξ belongs to the interior of $W + \Lambda$. Then, there exists a neighborhood of it contained in $W + \Lambda$, which contains a ξ' such that

$$\langle n, \xi' - \zeta \rangle > \langle n, \xi - \zeta \rangle = \langle n, \lambda \rangle ,$$

and since $\xi' \in W + \Lambda$, there exists a ζ' and a λ' such that,

$$\zeta' \in W ; \lambda' \in \Lambda \qquad \xi' = \zeta' + \lambda' .$$

We have

$$\langle n, \lambda' \rangle = \langle n, \xi' - \zeta' \rangle = \langle n, \xi' - \zeta \rangle + \langle n, \zeta - \zeta' \rangle .$$

Now, because W is convex and n is a normal, it is an elementary property of convex sets that

$$\langle n, \zeta - \zeta' \rangle \geq 0$$

and consequently

$$\langle n, \lambda' \rangle \geq \langle n, \xi' - \zeta \rangle > \langle n, \lambda \rangle$$

which is in contradiction with the definition of λ. This proves the direct part. Conversely, let n be a normal to $W + \Lambda$ at ξ. Then $\langle n, \xi \rangle$ is maximum, and thus both $\langle n, \lambda \rangle$ and $\langle n, \zeta \rangle$ are. Hence n is normal to W at ζ, and the lemma is proved.

From this fact, we infer a simple characterization of the strategy that verifies (2.4) under condition (2.5).

Let n_o be a normal to W_{τ_o} at $\zeta_o = \pi\Phi(\tau_o)z_o$. Any $v^o(\cdot)$ verifying

$$\langle n_o, \pi \int_{\tau_o - \epsilon}^{\tau_o} \Phi(r) v^o(\tau_o - r) dr \rangle = \text{maximum}$$

verifies (2.4), i.e., since the inner product is linear

$$\int_{\tau_o - \epsilon}^{\tau_o} \langle n_o, \pi\Phi(r) v^o(\tau_o - r) \rangle dr = \text{maximum} .$$

Thus, v^o must verify almost everywhere in $r \in [\tau_o-\epsilon,\tau_o]$

$$\langle n_o, \pi\Phi(r)v^o(\tau_o-r)\rangle = \max_{v \in Q} .$$

This implies, since the operator $\pi\Phi(r)$ is continuous, that $v^o(t)$ belongs to the boundary of Q for almost every t. This is a game theoretic version of Contensou's optimality principle. For instances of its use in differential games, see [18] and [20].

Another important consequence of (2.6) is that the function $v^o(\cdot)$ is actually independent of ϵ. Consequently, its limit as ϵ goes to zero is simply its limit as t goes to zero, and is given by

$$v^* = \text{Arg} \max_{v \in Q} \langle n_o, \pi\Phi(\tau_o)v\rangle . \tag{2.6}$$

Similarly, whatever $v(\cdot)$ is there exists a normal n_ϵ to $W_{\tau_1-\epsilon}$ such that u^o verifies

$$\langle n_\epsilon, \pi\Phi(r)u^o(\tau_1-r)\rangle = \max . \tag{2.6a}$$

Notice that if $v = v^o$, then $n_\epsilon = n_o$.

The function $u^o(\cdot)$ depends on ϵ and $v[0,\epsilon]$ through n_ϵ. However, as ϵ goes to zero, and consequently τ_1 to t_o, n_ϵ tends to a normal n_o to W_{τ_o}, and $u^o(0)$ to an argument of the corresponding maximum:

$$u^* = \text{Arg} \max_{u \in P} \langle n_o, \pi\Phi(\tau_o)u\rangle .$$

In particular, if n_o is unique, this is independent of v, and uniquely defined if P is strictly convex.

2.6 Technical Results

At this point, we need some technical results. We consider the space K of the convex compact subsets of L, and, following Pontryagin [24], give it a metric defined by

$$\text{dist}(A,B) = \max_{a\in A\ b\in B} \{\text{dist}(a,B) , \text{dist}(b,A)\} .$$

It is easy to check that this is indeed a distance. It is stated in [24] that with the induced topology, K is complete.

We also introduce the notation $\zeta(\tau) = \pi\Phi(\tau)z_o$, $T(z_o) = \tau_o$ and for τ smaller than τ_o let the distance from $\zeta(\tau)$ to W_τ be $D(\tau)$:

$$D(\tau) = \text{dist}\left(\zeta(\tau), W_\tau\right) = \min_{\eta\in W_\tau} \|\zeta(\tau)-\eta\| = \|\zeta(\tau)-\eta(\tau)\| \;.$$

$\eta(\tau)$ is uniquely defined in W_τ , due to the convexity of this set.

<u>Lemma 1</u>: The vector $\eta(\tau)$ is left differentiable at $\tau = \tau_o$.

<u>Proof</u>: Let

$$\Delta\zeta(\delta\tau) \stackrel{\Delta}{=} \frac{1}{\delta\tau}\left[\zeta(\tau_o-\delta\tau)-\zeta_o\right] \qquad \zeta_o \stackrel{\Delta}{=} \zeta(\tau_o)$$

$$\Delta\eta(\delta\tau) \stackrel{\Delta}{=} \frac{1}{\delta\tau}\left[\eta(\tau_o-\delta\tau)-\eta_o\right] \qquad \eta_o \stackrel{\Delta}{=} \eta(\tau_o) = \zeta_o$$

and

$$\Delta W(\delta\tau) \stackrel{\Delta}{=} \frac{1}{\delta\tau}\left[W_{\tau_o-\delta\tau}-\zeta_o\right] \qquad \text{is a convex set.}$$

Clearly, $\Delta\zeta(\delta\tau)$ has a limit $\Delta\zeta = -\pi\Phi(\tau_o)Cz_o$ as $\delta\tau$ goes to zero and

$$\Delta\zeta(\delta\tau) = \dot{\Delta\zeta}_o + 0(\delta\tau) \;.$$

We also have, since $\eta(\tau)$ belongs to W_τ ,

$$\Delta\eta(\delta\tau) \in \Delta W(\delta\tau) = \left(\frac{1}{\delta\tau}\,(W_{\tau_o}-\zeta_o) + \frac{1}{\delta\tau}\int_{\tau_o-\delta\tau}^{\tau_o} Q_r dr\right)$$

$$\stackrel{*}{-}\frac{1}{\delta\tau}\int_{\tau_o-\delta\tau}^{\tau_o} P_r dr \;.$$

This expression for ΔW is valid because W_τ verifies (2.5).

Let K_{τ_o} be the tangent cone to $W_{\tau_o} - \zeta_o$ at the origin. Locally, the boundary of $W_{\tau_o} - \zeta_o$ is contained between K_{τ_o} and an arbitrary, fixed cone interior to K_{τ_o} . Thus, locally, as $\delta\tau$ goes to zero, $\frac{1}{\delta\tau}\,(W_{\tau_o}-\zeta_o)$ can be made arbitrarily close

to K_{τ_o} (which is invariant under scalar multiplication). Also, the integrals can be made arbitrarily close to Q_{τ_o} and P_{τ_o}. Therefore, $\Delta W(\delta\tau)$ has, locally, a limit ΔW_o :

$$\Delta W_o = \left(K_{\tau_o} + Q_{\tau_o}\right) \pm P_{\tau_o} .$$

Let $\Delta\eta_o$ be the closest point of ΔW_o to $\Delta\zeta_o$. We shall prove that it is the limit of $\Delta\eta$:

$$\Delta\eta(\delta\tau) \to \Delta\eta_o \qquad \text{as} \qquad \delta\tau \to 0 .$$

Assume the above statement is false. Then, there exists an ϵ^* positive such that for every $\delta\tau$, there is a $\delta\tau_1$ smaller than $\delta\tau$, for which

$$\Delta\eta(\delta\tau_1) = \Delta\eta_1 \qquad \|\Delta\eta_1 - \Delta\eta_o\| > \epsilon^* .$$

Replacing ΔW_o by its tangent plane at $\Delta\eta_o$, it is easy to see that this implies that there exists a fixed β such that

$$\|\Delta\zeta_o - \Delta\eta_1\| > \|\Delta\zeta_o - \Delta\eta_o\| + \beta . \tag{*}$$

Now, in view of what was said previously, we can choose $\delta\tau$ such that for any $\delta\tau_1 < \delta\tau$, with $\Delta\zeta(\delta\tau_1) = \Delta\zeta_1$,

$$\|\Delta\zeta_o - \Delta\zeta_1\| < \alpha$$

$$\mathrm{dist}(\Delta W_1, \Delta W_o) < \alpha \qquad \text{so that} \qquad \mathrm{dist}(\Delta\eta_o, \Delta W_\tau) < \alpha$$

then, since $\Delta\eta_1$ provides a minimum of $\|\Delta\zeta_1 - \Delta\eta\|$, $\Delta\eta \in \Delta W(\delta\tau)$, and with the last inequality, and the triangular inequality:

$$\|\Delta\zeta_1 - \Delta\eta_1\| < \|\Delta\zeta_1 - \Delta\eta_o\| + \alpha$$

and therefore, with the first inequality

$$\|\Delta\zeta_o - \Delta\eta_1\| < \|\Delta\zeta_o - \Delta\eta_o\| + 2\alpha .$$

Now β is a fixed number independent of $\delta\tau$. Thus α can be chosen smaller than $\frac{1}{2}\beta$, and this gives a contradiction with the inequality (*). Therefore $\Delta\eta(\delta\tau)$ has a limit $\Delta\eta_o$, and we have

$$\left.\frac{d\eta(\tau)}{d\tau}\right|_{\tau=\tau_o} = -\Delta\eta_o$$

and Lemma 1 is proved.

Lemma 2: The convergence of $\Delta\eta$ to $\Delta\eta_o$ is Lipschitz in τ.

Proof: The proof is elementary and cumbersome. It presents no interest in itself and will only be sketched.

Consider first the closest point $\Delta\eta^*$ of ΔW_τ to $\Delta\zeta_o$. It is sufficient to prove the result for the convergence of $\Delta\eta^*$ to $\Delta\eta_o$, and then remark that by the convexity of ΔW_τ, $\Delta\eta^*$ is closer to $\Delta\eta$ than $\Delta\zeta$ to $\Delta\zeta_o$.

To prove the result for $\Delta\eta^*$, notice that the convergence of ΔW_τ to ΔW_o is Lipschitz, from the argument of the fixed cones presented in Lemma 1.

Then distinguish between two cases:

- $\Delta\eta_o$ is not a corner point. Then show that the directions of the normals to ΔW_τ at $\Delta\eta^*$ and to ΔW_o at $\Delta\eta_o$ must agree to first order, which gives the desired result.
- $\Delta\eta_o$ is a corner point. Then the result comes from simple geometric arguments on the farthest point where $\Delta\eta^*$ can be, knowing that the boundary of ΔW_τ lies within first order distance of the boundary of ΔW_o.

In both cases, the result is proved.

Corollary 1: Let $n(\tau) = \zeta(\tau) - \eta(\tau)$, $\hat{n}(\tau) = \frac{n(\tau)}{\|n(\tau)\|}$; then

$$\dot{n}_o = \left.\frac{dn(\tau)}{d\tau}\right|_{\tau=\tau_o} = \Delta\eta_o - \Delta\zeta_o ,$$

and if $\dot{n}_o \neq 0$, $\hat{n}(\tau)$ has a limit $\hat{n}_o$, and is Lipshitz, as $\delta\tau \to 0$.

Proof: The first part of the claim is trivial.

Because of Lemmas 1 and 2,

$$n(\tau) = -\dot{n}_o \, \delta\tau + \delta\tau \, \vec{O}(\delta\tau)$$

where $\vec{O}(\delta\tau)$ is a vector verifying $\|\vec{O}(\delta\tau)\| = O(\delta\tau) \leq p\, \delta\tau$. Thus

$$\|n(\tau)\| = \|\dot{n}_o\|\delta\tau + \delta\tau \, O(\delta\tau)$$

and

$$\hat{n}(\tau) = -\frac{\dot{n}_o}{\|\dot{n}_o\|} + \frac{\dot{n}_o}{\|\dot{n}_o\|} O(\delta\tau) + \vec{O}(\delta\tau)$$

which proves the claim, with

$$\hat{n}_o = -\frac{\dot{n}_o}{\|\dot{n}_o\|} .$$

<u>Corollary 2</u>: The distance $D(\tau)$ is left differentiable at τ_o, its derivative is

$$-\frac{dD(\tau)}{d\tau}\bigg|_{\tau=\tau_o} = \Delta D_o = \|\dot{n}_o\| .$$

<u>Proof</u>: We have

$$D(\tau) = \|n(\tau)\| , \quad \text{and} \quad \Delta D(\delta\tau) = \frac{D(\tau)-D(\tau_o)}{\delta\tau} = \frac{D(\tau)}{\delta\tau}$$

$$= \frac{\|n(\tau)\|}{\delta\tau}$$

which, with the calculation of Corollary 1, proves Corollary 2.

<u>Remark 1</u>: It is easy to see, by the separation theorem for convex sets, that $n(\tau)$ is a normal to W_τ. Consequently, if the normals are unique, Lemma 1 can be proved more directly. However, we want to allow for the case where ζ_o is a corner of W_{τ_o}, and we need Lemma 2.

<u>Remark 2</u>: A proof similar to the one we gave in Lemma 1, using the same tools, can easily be made to prove the convergence of n_ϵ to n_o in the previous section.

2.7 Sufficiency: A Local Theorem for ϵ-Strategies

Again, let z_o and τ_o be such that $T(z_o) = \tau_o$:

$$\zeta(\tau_o) = \pi\Phi(\tau_o)z_o \in \partial W_{\tau_o} \qquad \zeta(\tau) \not\in W_\tau \ , \quad \forall\tau < \tau_o \ .$$

Following Pshenichnyi [25], we define a <u>regular point</u> as a point z_o where

$$\frac{d}{d\tau}\left[\operatorname{dist}\left(\zeta(\tau), W_\tau\right)\right]_{\tau=\tau_o} = -\Delta D_o \neq 0 \ ,$$

and thus, this number is negative. We have seen in the previous section that this derivative exists.

<u>Theorem</u>: If z_o is a regular point, under condition (2.5) the evader can prevent a jump of the estimating function at z_o. Precisely, there exists an ϵ_o such that for any ϵ smaller than ϵ_o, $T\left(z(\epsilon)\right) = \tau_o - \epsilon$.

<u>Proof</u>: We want to prove that using v^o defined in Section 2.5 there does not exist any τ smaller than τ_o, $\delta\tau = \tau_o - \tau_1$, such that

$$\pi\Phi(\tau_1)z_o + \int_{\tau_1-\epsilon}^{\tau_1} \pi\Phi(r)v^o(\tau_1-\epsilon)dr \in W_{\tau_1-\epsilon} + \int_{\tau_1-\epsilon}^{\tau_1} \pi\Phi(r)Pdr$$

$$= W_{\tau_1} + \int_{\tau_1-\epsilon}^{\tau_1} \pi\Phi(r)Qdr \ . \tag{2.8}$$

First, notice that by definition of τ_o, $\delta\tau$ goes to zero as ϵ does, for any τ_1 that would verify (2.8).

For every given $s \in [0,\epsilon]$, let $r_1 = \tau_1 - s$ and $r_o = \tau_o - s$. Also, let v^1 provide the maximum in the following product (where $v_r = \pi\Phi(r)v$)

$$\max_{v \in Q} \langle v_{r_1}, \hat{n}(\tau_1)\rangle = \langle v^1_{r_1}, \hat{n}(\tau_1)\rangle$$

where $\hat{n}(\tau)$ is defined as in Section 2.6. Let also $\hat{n}(\tau_1) = \hat{n}_1$. We have, for every s,

$$\langle v^1_{r_1}, \hat{n}_1\rangle - \langle v^o_{r_1}, \hat{n}_1\rangle \leq |\langle v^1_{r_1}, \hat{n}_1\rangle - \langle v^o_{r_o}, \hat{n}_o\rangle| + |\langle v^o_{r_o}, \hat{n}_o\rangle - \langle v^o_{r_o}, \hat{n}_1\rangle| + |\langle v^o_{r_o}, \hat{n}_1\rangle - \langle v^o_{r_1}, \hat{n}_1\rangle|$$

and we show that the three differences on the right hand side are Lipschitz in $\delta\tau$. It is obvious for the third one, and it is a consequence of Lemma 2 in the previous section for the second one.

The first difference is the variation of the support function of W_τ :

$$\delta^*\left(\hat{n}(\tau)\,|\,W_\tau\right) .$$

It is proved in [26] that δ^* is convex in $\hat{n}$, and, as such, Lipschitz at any point of the relative interior of its domain. This, together with Lemma 2 and the obvious Lipschitz character of its dependence on W_τ, proves that this first difference has the claimed property. Consequently, there exists an M independent of $\delta\tau$ such that

$$\langle v^1_{r_1}, \hat{n}_1\rangle - \langle v^o_{r_1}, \hat{n}_1\rangle \leq M\delta\tau \quad \forall s . \tag{2.9}$$

Moreover, remember that $v^o(s)$, and thus $\hat{n}_1$ and v^1, are not functions of ϵ. Thus M can be chosen independent of ϵ as well.

We integrate the previous relation for s varying from 0 to ϵ :

$$\left\langle \int_{\tau_1-\epsilon}^{\tau_1} [v^1_r(\tau_1-r) - v^o_r(\tau_1-r)]dr \;,\; \hat{n}_1\right\rangle \leq M\,\epsilon\,\delta\tau .$$

We subtract $\eta(\tau_1)$ (defined as in Section 2.6) from the left hand side of (2.8), and project on $\hat{n}_1$, a normal to W_{τ_1} at $\eta(\tau_1)$, and compare with the right hand side, using the lemma of Section 2.4. This gives

$$\langle n(\tau_1) + \int_{\tau_1-\epsilon}^{\tau_1} v_r^o(\tau_1-r)dr \ , \ \hat{n}(\tau_1)\rangle$$

$$- \max_{v(\cdot)} \langle \int_{\tau_1-\epsilon}^{\tau_1} v_r(\tau_1-r)dr \ , \ \hat{n}(\tau_1)\rangle$$

$$= D(\tau_1) - \langle \int_{\tau_1-\epsilon}^{\tau_1} [v_r'(\tau_1-r) - v_r^o(\tau_1-r)]dr \ , \ \hat{n}_1\rangle$$

$$\geq \Delta D_o \delta\tau + O(\delta\tau^2) - M\epsilon\delta\tau \ .$$

But ϵ can be chosen to make this difference positive for every $\delta\tau$, since $\delta\tau$ goes to zero with ϵ, which proves that inclusion (2.8) is not verified. This proves the theorem.

Remark: Lemma 2 allows us to use the intermediary of the function δ^*, and thus avoids an investigation into the regularity of the function v.

Under condition (2.5) we have not only proved that the variation of $T(z)$ is locally optimal, but also we have the much stronger result that there exists an ϵ-strategy actually yielding this rate of decrease. This may be considered as very important when it comes to the implementation of an optimal control.

2.8 Sufficiency: A Local Theorem for the Limit Process

In the previous section, we were seeking an ϵ-strategy yielding the time of capture $T(z)$. But the existence of such a strategy is not necessary for $T(z)$ to be optimal. It suffices to be able to find δ-efficient strategies for arbitrarily small δ's. Then, according to our discussion of Section (2.3), $T(z)$ is optimal and corresponds to a saddle point.

Consider a trajectory $z(t)$, and let

$$z_\tau = \pi\Phi(\tau_o - t)z(t) \qquad z_{\tau_o} = \zeta_o \in \partial W_{\tau_o} \qquad T\big(z(0)\big) = \tau_o$$

and notice that, with $\tau_o - t = \tau$,

$$-\frac{d}{d\tau} z_\tau \Big|_{\tau_o} = \lim_{\delta\tau\to 0}\left[\frac{1}{\delta\tau}\left(z_{\tau_o-\delta\tau}-z_{\tau_o}\right)\right] = v_{\tau_o} - u_{\tau_o} .$$

Assume that, for every τ , $z_\tau \in W_\tau$ $\tau_o-\epsilon < \tau \leq \tau_o$.

We have:

$$\Delta z_\tau(\delta\tau) = \frac{1}{\delta\tau}(z_\tau - z_{\tau_o}) \in \frac{1}{\delta\tau}(W_\tau - z_{\tau_o}) = \Delta W(\delta\tau) .$$

It is easy to check that $\mathrm{dist}[\Delta W(\delta\tau+\epsilon),\Delta W(\delta\tau)]$ goes to zero with ϵ . The space K being complete, there exists a limit

$$\Delta W_o = \lim_{\delta\tau\to 0}\frac{1}{\delta\tau}\left(W_{\tau_o-\delta\tau}-\zeta_o\right) ,$$

so that, in particular, our Corollary 2, Section 2.6, holds without condition (2.5). The above inclusion yields

$$v_{\tau_o} - u_{\tau_o} \in \Delta W_o .$$

With our definition of W_τ , the pursuer can always achieve this goal. Thus

$$(\Delta W_o + P_{\tau_o}) \subset Q_{\tau_o} \qquad \text{or} \qquad 0 \in (\Delta W_o + P_{\tau_o}) \stackrel{*}{-} Q_{\tau_o} .$$

With these definitions and remarks, we prove the following fact:

<u>Theorem</u>: At a regular point z_o , if the origin belongs to the boundary of $(\Delta W_o + P_{\tau_o}) \stackrel{*}{-} Q_{\tau_o}$, the evader can insure that the estimating function will have a jump of the order of $\epsilon\, O(\epsilon)$ at most.

<u>Proof</u>: Under the conditions of the theorem, there exists a v^* such that $v^*_{\tau_o} - u_{\tau_o}$ cannot belong to the interior of ΔW_o .

If the pursuer does not choose $u^*_{\tau_o}$ such that $v^*_{\tau_o} \in \partial\Delta W_o$, according to our previous calculations z_τ cannot belong to W_τ for every τ in a neighborhood of τ_o . Therefore, let us assume that $u = u^*$. Let

$$v^*_{\tau_o} - u^*_{\tau_o} = \Delta y_o \in \partial\Delta W_o .$$

Δy_o is the limit of a function Δy_τ belonging to ΔW_τ , of the form

$$\Delta y_\tau = \frac{1}{\delta\tau}(y_\tau - \zeta_o) \in \Delta W(\delta\tau)$$

$$y_\tau \in W_\tau .$$

Moreover, y_τ can be chosen to belong to the boundary of W_τ. Let z^*_τ be a trajectory generated by a strategy (u,v) agreeing with u^*,v^* at τ_o. For instance, u and v constant. Then

$$\frac{1}{\delta\tau}(z^*_\tau - \zeta_o) = \frac{1}{\delta\tau}(y_\tau - \zeta_o) + O(\delta\tau)$$

and thus

$$z^*_\tau = y_\tau + \delta\tau\, O(\delta\tau) ,$$

which proves that z^*_τ is at a distance $\delta\tau\, O(\delta\tau)$ of the boundary of $W_{\tau_o - \delta\tau}$ at most. Using the assumption that z_o is regular, ΔD_o is defined and non-zero. The jump of the estimating function is of the order of

$$\frac{1}{\Delta D_o} \operatorname{dist}(z_\tau, \partial W_\tau) ,$$

and thus, with a given step ϵ, this jump is of the order $\epsilon\, O(\epsilon)$ at most. This proves the theorem.

Assume, now, that this holds in the neighborhood of a trajectory, except, possibly, at finitely many points on any trajectory. Then, if we decrease ϵ, the number of steps in a given interval increases like $1/\epsilon$. But if the jumps decrease as $\epsilon\, O(\epsilon)$, the total jump during that time goes to zero with ϵ. We say that, locally, we have exhibited a δ-efficient strategy for arbitrary δ.

What remains to be done is to see whether the set family W_τ has the properties required by the theorem. We shall prove, in the next section, that Pontryagin's alternating integral verifies the following relation.

<u>Proposition:</u>

$$K_{\tau_o} = (\Delta W_o + P_{\tau_o}) \stackrel{*}{-} Q_{\tau_o}$$

where K_{τ_o} is the tangent cone to $W_{\tau_o} - \zeta_o$ at the origin. Since $0 \in \partial K_{\tau_o}$, our previous discussion holds.

2.9 Properties of the Alternating Integral

To carry out our program, we need some preliminary definitions and results.

Definition: A geometric difference $C = A \stackrel{*}{-} B$ is said to have complete sweeping (c.s.) in the direction of n when a boundary point of $C + B$ having this direction for normal is also a boundary point of A.

Notice that then all such points will have that property. Notice also that at every boundary point of C there is at least one normal having c.s. in its direction.

Lemma: For every set A, B and C for which this combination exists, we have

$$[(A+B)\stackrel{*}{-}C)+B)\stackrel{*}{-}C = (A+2B) \stackrel{*}{-} 2C$$

where the notation of the left hand side has an obvious meaning.

Proof: From the results of Section 1.6, we would have the left hand side included in, or equal to, the right hand side. To prove the equality, we prove that any boundary point of the left hand side is a boundary point of the right hand side.

To do so, we prove that all three geometric differences have complete sweeping in any direction in which the last one of the left hand side has. Once this is proved, the result follows rapidly:

Let

$$D_1 = (A+B) \stackrel{*}{-} C \qquad D_2 = (D_1+B) \stackrel{*}{-} C \qquad D = (A+2B) \stackrel{*}{-} 2C .$$

Consider a boundary point d_2 of D_2, and a normal n to D_2 at d_2 such that $(D_1+B) \stackrel{*}{-} C$ has c.s. in its direction. Consider then

$$b = \operatorname{Arg} \max_{b \in B} \langle n,b \rangle$$

$$c = \operatorname{Arg} \max_{c \in C} \langle n,c \rangle \ .$$

Because of the hypothesis of c.s. in that direction, we have

$$d_2 + c \in \partial(D_1+B)$$

and as n is also normal to $D_1 + B$ at $d_2 + c$,

$$d_1 = d_2 + c - b \in \partial D \ , \quad n \text{ normal to } D_1 \text{ at } d_1 \ .$$

By the same reasoning, we deduce that

$$d_1 + c - b \in \partial A$$

with $d_1 = d_2 + 2c - 2b$, and again using the fact that n is a common normal, that this implies

$$d_2 \in \partial[(A+2B) \stackrel{*}{-} 2C] \ .$$

Therefore, the only thing we have left to prove is the following proposition:

<u>Proposition</u>: Let n be a direction in which $(D_1+B) \stackrel{*}{-} C$ has c.s., then $(A+B) \stackrel{*}{-} C$ and $(A+2B) \stackrel{*}{-} 2C$ have c.s. in that direction.

<u>Proof</u>: If we replace the first set of a geometric difference by a set which has at every corresponding point of its boundary (common normal) a bigger radius of curvature of "less acute" corner points (larger cone of normals), no direction can lose its c.s. property.

$[(A+2B) \stackrel{*}{-} C]$ has this relationship with $[\left((A+B) \stackrel{*}{-} C\right) + B]$, thus, in a direction n where $(D_1+B) \stackrel{*}{-} C$ has c.s., $[(A+2B) \stackrel{*}{-} C] \stackrel{*}{-} C$ has, too. Now, notice that (see Section 1.6) $[(A+2B) \stackrel{*}{-} C] \stackrel{*}{-} C = (A+2B) \stackrel{*}{-} 2C$. Take a boundary point d of this set, where n is a normal. Let $c \in C$ maximize the inner product $\langle n,c \rangle$. Then

$$d + c \in \partial[(A+2B) \stackrel{*}{-} C] \ ,$$

and comparing the normals, it follows that

$$d + 2c \in \partial(A+2B) \ .$$

Thus, $(A+2B) \stackrel{*}{-} 2C$ has c.s. in the direction of n. Then, because of our introductory remark, $(2A+2B) \stackrel{*}{-} 2C$ has, too. And by mere similitude, $(A+B) \stackrel{*}{-} C$ as well.

Thus, the proposition is proved, and, consequently, the lemma.

Now, we can prove the last proposition of the previous section. Notice that, by induction, this property is true for any alternating sum of two sets of the form

$$\left[A+B) \stackrel{*}{-} C) + B) \stackrel{*}{-} C....) + B) \stackrel{*}{-} C = (A+nB) \stackrel{*}{-} nC \qquad n = 2^p \ .$$

Consider the set W_{τ_o} defined by the alternating integral

$$W_{\tau_o} = \int_{e,o}^{\tau_o} [P_r \stackrel{*}{-} Q_r] dr \ .$$

We have

$$\frac{1}{\delta\tau} [W_{\tau_o} - \zeta_o] \subseteq \left(\frac{1}{\delta\tau} (W_{\tau_o - \delta\tau} - \zeta_o) + \frac{1}{\delta\tau} \int_{\tau_o - \delta\tau}^{\tau_o} P_r dr \right)$$

$$\stackrel{*}{-} \frac{1}{\delta\tau} \int_{\tau_o - \delta\tau}^{\tau_o} Q_r dr \ .$$

The left hand side can be made arbitrarily close to

$$\left[\frac{1}{\delta\tau} \left(W_{\tau_o - \delta\tau} - \zeta_o \right) + \frac{1}{\delta\tau} \int_{\tau_o - \delta\tau}^{\tau_o - \frac{n-1}{n}\delta\tau} P_r dr \right) \stackrel{*}{-} \frac{1}{\delta\tau} \int_{\tau_o - \delta\tau}^{\tau_o - \frac{n-1}{n}} Q_r dr \Big) .... \Big)$$

$$+ \frac{1}{\delta\tau} \int_{\tau_o - \frac{1}{n}\delta\tau}^{\tau_o} P_r dr \Big) \stackrel{*}{-} \frac{1}{\delta\tau} \int_{\tau_o - \frac{1}{n}\delta\tau}^{\tau_o} Q_r dr \qquad n = 2^p$$

which, by continuity of the geometric difference for convex sets (see [24]) can be made arbitrarily close to

$$\left[\frac{1}{\delta\tau}\left(W_{\tau_o-\delta\tau}-\zeta_o\right)+\frac{1}{n}P_{\tau_o}\right)\overset{*}{-}\frac{1}{n}Q_{\tau_o}\Big)+\frac{1}{n}P_{\tau_o}\Big)\overset{*}{-}\frac{1}{n}Q_{\tau_o}\Big)\cdots$$

$$=\left[\frac{1}{\delta\tau}\left(W_{\tau_o-\delta\tau}-\zeta_o\right)+P_{\tau_o}\right]\overset{*}{-}Q_{\tau_o}\ .$$

This set, in turn, can be made arbitrarily close to the right hand side of the above inclusion. Thus, the distance between the two sides of this inclusion goes to zero with $\delta\tau$, and we have the property claimed:

$$K_{\tau_o}=(\Delta W_o+P_{\tau_o})\overset{*}{-}Q_{\tau_o}\ ,$$

the existence of the other limits proving that $\Delta W_o \neq E$. Notice that ΔW_o is not necessarily a cone, but has K_{τ_o} as its recession cone. The above geometric difference does not necessarily have complete sweeping. It has under condition (2.5) as we saw in Section 2.6.

2.10 Sufficiency: A Global Condition

In this section, we shall assume without proof that the strategies u^* and v^* only have isolated simple jumps, so that there always exists a left continuous definition of them at any point.

Should this be not true at some point, only the strong version of the condition derived would hold, and it would no longer imply the weak one. Notice that if such a behavior happened at more than isolated points of a trajectory, we could always replace the "chattering" control by an equivalent non-chattering one, due to the convexity of the control sets.

i) The Problem. We have seen that under condition (2.5), if $T(z_o)=\tau_o$, there exists an ϵ^o such that $T\,z(\epsilon^o)=\tau_o-\epsilon^o$. Let $z(\epsilon^o)=z'$, $T(z')=\tau_o-\epsilon^o$, and there exists an ϵ' having the same property, etc. However, what may happen is that

$$\sum_{i=0}^{\infty}\epsilon^i=e<\tau_o \qquad \tau_o-e=\tau^o>0\ .$$

Then, the point $z^o=z(e)$ is such that

$$\pi\Phi(\tau^o)z^o \in \partial W_{\tau^o}$$

$$\pi\Phi(\tau')z^o \in \partial W_{\tau'} \qquad \tau^o - \tau' = \delta\tau > 0 \tag{2.10}$$

or a non-regular point. Otherwise our local proofs would hold in a neighborhood of z^o, in contradiction with the hypothesis. If we rule out, by assumption, non-regular points of second kind (see next section), then, as we shall see in the next proposition, relations (2.10) hold at z^o.

Similarly for the case of the continuous process, we have seen that a jump would be of the order of $\epsilon\, O(\epsilon)$ because the distance of z_τ to the boundary of W_τ is of that order. But the proof fails if $z(t)$ comes arbitrarily close to a point verifying (2.10).

Therefore, we must impose some conditions on points of this type. And since z^o can be approached arbitrarily closely on the trajectory without jumps, the only points to consider are those of the following set F:

$$F = \partial\{[\Phi(-\tau^o)\pi^{-1}\partial W_{\tau^o}] \cap [\Phi(-\tau')\pi^{-1}\partial W_{\tau'}]\} \qquad \tau^o - \tau' = \delta\tau > 0$$

being understood that this boundary is to be considered only where it separates the intersection from a region where $T(z)$ is in the neighborhood of τ^o.

<u>Proposition</u>: F is the union of parts of the boundary of $\mathcal{C}$, and of loci of non-regular points of first kind.

<u>Proof</u>: We assume that $\pi\Phi(\tau')z^o \in \partial W_{\tau'}$, but an arbitrarily close point $z^{1}_{\tau^o}$ does not belong to any $W_{\tau'}$, with τ' in the neighborhood of τ'. This can happen in two ways:

- Either: for ϵ sufficiently small, locally we have

  $$\partial W_\tau \subset W_{\tau^o} \qquad \forall\tau \in (\tau^o-\epsilon, \tau^o+\epsilon)\ .$$

  Then it is easy to see that z^o is non-regular at τ'; moreover, $z^o_{\tau'}$ belongs to the envelope of the W_τ's, which will be seen to be the characteristic property of non-regular points of first kind.

- Or else: W_τ is not defined in an open neighborhood of τ'. But

if it is defined and has an interior for τ', it is defined, by continuity for some larger τ. The only possibility, thus, is that $\tau' = 0$ so that W_τ is not defined for $\tau < \tau'$. Then $W_{\tau'} = \mathcal{C}$.

Therefore, the problem is reduced to checking whether the trajectories we have defined penetrate such surfaces. In the absence of non-regular points of second kind, we have the following result:

Theorem: The necessary and sufficient condition for the estimating function to be optimal is that the corresponding trajectories do not cross the manifold F.

ii) Sufficient Conditions. Sufficient conditions can be derived on the structure of the game, not requiring the actual computation of the trajectories. Notice that, in principle, once the sets W_τ are known, the manifold F is known.

A first, obvious, sufficient condition is $F = \emptyset$. However, this rarely happens, although one could construct examples that satisfy this condition.

We can deduce different conditions from another idea: it suffices to insure that a trajectory arriving at z^o would lie in $V_{\tau'-\epsilon}$ a time ϵ earlier. Then, there is no jump in $T(z)$ at z^o; thus the problem mentioned does not occur. This is what we shall call "condition B." It is insured by the following strong version:

Let $z^o \in F$; there exists a neighborhood of z^o for which, if $z_{\tau^o+\epsilon} \in \partial W_{\tau^o+\epsilon}$, there is a normal n_ϵ to $W_{\tau^o+\epsilon}$ at $z_{\tau^o+\epsilon}$ such that the corresponding u^o and v^o verify, for every normal n' to $W_{\tau'}$ at $z^o_{\tau'}$

$$\begin{aligned} &\langle n', v^o_\tau(s) - u^o_\tau(s)\rangle \geq \langle n', v'_\tau(s) - u'_\tau(s)\rangle \\ &\forall\tau = \tau' + \epsilon - s \qquad 0 \leq s \leq \epsilon \end{aligned} \tag{2.11}$$

where $u'(\cdot)$ and $v'(\cdot)$ are defined similarly to u^o, v^o, with n' and τ'. This condition, directly derived from

$$\pi\Phi(\tau'+\epsilon)\, z(t_o-\epsilon) \in W_{\tau'+\epsilon},$$

actually means that, judged according to the sets $V_{\tau'}$, the strategies (u^o,v^o) are not worse, for the evader, than the optimal pair (u',v') .

This condition is still complicated, but two interesting forms can be derived from it, easier to check.

The first one is condition A of Gusyatnikov and Nikolsky (see Section 2.2). In that case, a u^o corresponds to v^o such that

$$u^o_\tau - v^o_\tau \in P_\tau \overset{*}{-} Q_\tau \qquad \forall\tau$$

insuring that condition B is satisfied.

The superiority of this condition is that it comes the closest to dealing with the raw data of the problem. This point is investigated in more detail in [16]. Its main restriction is that it requires $P_\tau \not\supset Q_\tau$ for every τ , which gives the pursuer an excessive superiority over the evader.

Another form is the weak version of (2.11), valid with our assumption on the regularity of the optimal strategies. Then (2.12) is insured by

$$\langle n',v^*_{\tau'},-u^*_{\tau'}\rangle > \langle n',v'_{\tau'},-u'_{\tau'}\rangle \qquad (2.11)$$

which can be derived as a limit of (2.11) or by an argument similar to that of Section 2.8. Notice that if we allow "safe contact," then the strict inequality in (2.11) can be replaced by "greater than or equal to."

Finally, this is verified if v^* can be determined as a function of z only, independent of τ . Then, $v^* = v'$, and as u' provides a minimum in the expressions of (2.12) (or (2.11)), condition B is satisfied. This form is also a structural condition, not requiring that F be explicitly found.

2.11 Non-Regular Points

i) Characterization. Since we have been obliged to assume that all the points of our trajectories were regular, it is interesting to see in more detail what happens at a non-regular point. (We prefer to keep the

terminology singular for another type of point we shall introduce in Chapter 5.)

A non-regular point, we recall, is a point where

$$\frac{d}{d\tau}\operatorname{dist}\Big(\zeta(\tau),W_\tau\Big)\Big|_{\tau=\tau_o} = 0 \qquad \zeta(\tau_o) = \pi\Phi(\tau_o)z_o \in \partial W_{\tau_o}$$

$$\zeta(\tau) \not\in W_\tau \ , \ \forall\tau < \tau_o$$

We can easily verify the following fact:

<u>Proposition:</u> At a non-regular point, the gradient of the estimating function is infinite.

<u>Proof:</u> Consider an inverse image $z'(\tau)$ of $\eta(\tau)$ by

$$z'(\tau) \in \Phi(-\tau)\pi^{-1}\eta(\tau) \qquad T\Big(z'(\tau)\Big) = \tau \ .$$

It is possible to choose it in such a way that the limit of the line (z_o,z') does not reduce to ζ_o when acted upon by the operator $\pi\Phi(\tau_o)$. Thus, the length of $h(\tau) = z' - z_o$ verifies

$$\|h(\tau)\| = M(\tau)D(\tau) \quad \text{with} \quad M(\tau) \leq M_o \quad \text{as} \quad \tau \to \tau_o \ .$$

Now, as h goes to zero, we have

$$T(z) - T(z') = \delta\tau = \langle\nabla T(z),h\rangle + O(\delta\tau^2)$$

where $\nabla T(z)$ is the gradient of $T(z)$. Now, let $\hat{h} = \frac{h}{\|h\|}$,

$$\langle\nabla T(z),\hat{h}\rangle = \frac{\delta\tau+O(\delta\tau^2)}{M(\tau)\ D(\tau)} \geq \frac{1}{\Delta D_o M_o} + O(\delta\tau) \ .$$

Thus, if ΔD_o is zero, the inner product is infinite, which proves the proposition.

ii) <u>Classification</u>. To go further in the analysis, it is convenient to distinguish between two kinds of non-regular points:

- First kind: $\zeta(\tau)$ does not penetrate W_τ , and more precisely, there exists a positive ϵ such that

$$\zeta(\tau) \not\in W_\tau \qquad \forall\tau \in (\tau_o-\epsilon,\tau_o+\epsilon) \qquad \tau \neq \tau_o \ .$$

- Second kind: the above property is not verified. $\zeta(\tau)$ may either belong to the boundary of W_τ for a finite interval in τ , or penetrate into the interior of W_τ .

The interest of this classification appears in the following fact:

Proposition: A non-regular point of first kind lies on the envelope of the sets V_τ .

Proof: This immediately follows from the definition, transformed in terms of z_o and V_τ .

This envelope is clearly a discontinuity in the function $T(z)$. This is consistent with our remark that the gradient of $T(z)$ is infinite. Actually, the envelope is a barrier according to Isaacs. What we have here is a mere statement of Isaacs' envelope principle (see [18]).

iii) Properties of Barriers. Such an envelope is a closed manifold (see also [25]). Thus, if a trajectory reaches it from regular points, the set of regular points on this trajectory is open.

Moreover, if this trajectory comes from "outside" the barrier, namely, from the exterior of the union of sets, the envelope of which is the barrier, then it has a finite jump in the estimating function.

But because of the previous remark, the proof of Section 2.8 holds along the trajectory, yielding the following result:

Theorem: Under condition B, trajectories generated by the limit process never cross a locus of non-regular points of first kind that would induce a jump in the estimating function.

Finally, we have the following result, not really needed in a theory of the optimality of the process, but interesting because it corresponds to the cases which are usually met:

Theorem: If along a barrier as defined in this section the normal to W_τ is unique, then a trajectory generated by the limit process having a point in the barrier lies completely in it.

Proof: At a point where the normal to W_{τ_o} is unique, the cone K_{τ_o} is a half space, and its boundary the hyperplane tangent to W_{τ_o} . Then

$\partial\Delta W_o$ is a hyperplane parallel to ∂K_{τ_o} .

The fact that the point is non-regular means that

$$\Delta D_o = \Delta\eta_o - \Delta\zeta_o = 0 ,$$

and since $\Delta\eta_o$ lies, by definition, on $\partial\Delta W_o$, this is true also of $\Delta\zeta_o$. Now, consider

$$\pi\Phi(\tau_o)\dot{z}(0) = \pi\Phi(\tau_o)(cz_o - u^* + v^*) = v^*_{\tau_o} - u^*_{\tau_o} - \Delta\zeta_o .$$

If v^* and u^* are chosen according to (2.7) and (2.7a), $v^*_{\tau_o} - u^*_{\tau_o}$ belongs to $\partial\Delta W_o$. As $\Delta\zeta_o$ does, too, we see that $\pi\Phi(\tau_o)\dot{z}(0)$ is either zero or parallel to ∂K_{τ_o} , and thus to ∂W_{τ_o} .

In both cases, this implies that $\dot{z}_o$ is parallel to ∂V_{τ_o} . As we know that z_o lies on the envelope of the family V_τ , we see that under the continuous law, z remains on this envelope.

This proves the theorem.

iv) Non-Regular Points of Second Kind. Non-regular points of second kind appear as points where the estimating function has an infinite gradient without being discontinuous.

We propose the following example, which shows that such points can exist, and gives some indication of what they actually represent.

Consider a two-dimensional game where the geometrical space is the whole state space; therefore $\pi = I$ the identity. Let the dynamics be defined by

$$C = \begin{pmatrix} -\alpha & -\omega \\ \omega & -\alpha \end{pmatrix} \qquad \alpha,\omega \text{ positive real numbers}$$

and $P = Q$, so that $P_\tau \overset{*}{-} Q_\tau = \{0\}$. Finally, let the capture set $\mathcal{C}$ be the disk centered at the point

$$x = 0 \qquad\qquad y = -\frac{\alpha}{\omega}$$

and of radius $\ell = \sqrt{\omega^2+\alpha^2}/\omega$ so that its boundary goes through the point (1,0) .

Since $P_\tau \,\sharp\, Q_\tau = \{0\}$, W_τ is constant and equal to $\mathcal{C}$: with the proposed strategies, the two players' actions cancel each other and the state follows the free dynamics of the system. The transition matrix of C is

$$\Phi(\tau) = \begin{pmatrix} e^{-\alpha\tau} \cos \omega\tau & -e^{-\alpha\tau} \sin \omega\tau \\ e^{-\alpha\tau} \sin \omega\tau & e^{-\alpha\tau} \cos \omega\tau \end{pmatrix}$$

so that for a fixed z , $\zeta(\tau) = \Phi(\tau)z$ describes a logarithmic spiral as τ varies.

The capture circle has been chosen such that it is the osculating circle to the spiral through that point at $\alpha = 1$, $y = 0$. As a consequence, this whole spiral, outside of $\mathcal{C}$, is a locus of non-regular points of second kind.

In fact, for a point

$$z = \begin{pmatrix} e^{\alpha s} \cos \omega s \\ -e^{\alpha s} \sin \omega s \end{pmatrix} ,$$

we have

$$\zeta(\tau) = \begin{pmatrix} e^{-\alpha(\tau-s)} \cos \omega(\tau-s) \\ e^{-\alpha(\tau-s)} \sin \omega(\tau-s) \end{pmatrix}$$

and thus

$$D(\tau) = \left[e^{-2\alpha(\tau-s)} + \frac{2\alpha}{\omega} e^{-\alpha(\tau-s)} \sin \omega(\tau-s) + \frac{\alpha^2}{\omega^2} \right]^{1/2} - \frac{\sqrt{\alpha^2+\omega^2}}{\omega} .$$

It is a simple matter to check that

1) for $\tau < s$ $D(\tau) > 0$, and $D(s) = 0$

2) for $\tau = s$ $\frac{dD(\tau)}{d\tau} = 0$

3) for $\tau = s$ $\frac{d^2D(\tau)}{d\tau^2} = 0$ and $\frac{d^3D(\tau)}{d\tau^3} = -\alpha\omega\sqrt{\alpha^2+\omega^2} < 0$,

establishing that

1) $T(z) = s$

2) z is a non-regular point

3) it is a non-regularity of second kind

which is what we wanted to show.

Notice that $T(z) = s$ proves that the gradient of the estimating function has a finite component tangent to the spiral. Since this gradient is actually infinite, it is normal to the spiral.

This finishes our discussion of non-regular points of second kind.

2.12 Conclusion

We have characterized directly the controls u^o and v^o , and found a direct construction of their limits u^* and v^* .

An interesting feature is that while u^o is, under some conditions, optimal against every v , it was found that its limit u^* often does not depend on v . This is true if the sets W_τ do not present corner points, except, possibly, for countably many values of τ .

Notice also that

$$\langle n, \pi\Phi(\tau)(v-u)\rangle = \left\langle \left(\pi\Phi(\tau)\right)^* n, v-u\right\rangle$$

where $\left(\pi\Phi(\tau)\right)^*$ is the adjoint operator to $\pi\Phi(\tau)$. Let, then, $\lambda = \left(\pi\Phi(\tau)\right)^*\hat{n}$, and it is seen that the controls u^* and v^* must be such that

$$\langle \lambda, \dot{z}(u^*, v^*) \rangle = \min_u \max_v \langle \lambda, \dot{z}(u,v) \rangle$$

which bears a close resemblance to the Pontryagin Maximum Principle. Because of the possibility of the occurrence of corner points in the sets W_τ, the variation of λ may be difficult to describe. λ may even be non-unique. And it is noteworthy that the geometric subtraction can introduce corners without any of the constituting sets having one (and still without violating condition (2.5)).

With regard to the question of the optimality of the process described, we have found that under condition (2.5) the time $T(z)$ can be optimal for an ϵ-strategy, and a fortiori, of course, for the limit process. But this condition is not needed for the limit process, and we have found that the alternating integral is, as far as local behavior is concerned, the optimal capture set.

However, the corresponding trajectories can still fail to be optimal by crossing a barrier or penetrating the "non-usable part" of the capture set. Actually, in all instances known, it is the second phenomenon that occurs. As will be seen in the second part, this leads to state constrained optimal trajectories along a "safe contact." Condition B is sufficient to prevent this from happening.

In addition, all sufficiency conditions must exclude non-regular points of second kind. Apart from that, all conditions are both necessary and sufficient.

3. MULTISTAGE GAMES

In this chapter, we consider multistage games, that is, games in which the system to be controlled is in discrete time, governed by a difference equation. We shall briefly discuss the discrete equivalent of system (1.1), and then turn to the system theoretic formulation, with unbounded controls, for which the present technique turns out to be particularly well adapted.

3.1 The Discrete Game

In a very classical way, the system (1.1) can be transformed into a discrete one, letting

$$z(n\epsilon) \triangleq z(n) \qquad e^{\epsilon C} = \Phi(\epsilon) \triangleq \Phi$$

yields

$$z(n+1) = \Phi z(n) - u(n) + v(n) \tag{3.1}$$

where

$$u(n) \in P \qquad v(n) \in Q .$$

P and Q are compact convex sets derived from the original one in a trivial way. Define

$$P_n = \pi\Phi^n P \qquad Q_n = \pi\Phi^n Q .$$

And as in the first chapter, consider the sets

$$W_n = \left(\left(\cdots\left(\left(\left(\mathcal{C}+P_o\right) \overset{*}{-} Q_o\right) + P_1\right) \overset{*}{-} Q_1 \cdots\right) + P_{n-1}\right) \overset{*}{-} Q_{n-1}$$

$$W_n^{(\infty)} = \left(\mathcal{C} + \sum_{i=0}^{n-1} P_i\right) \overset{*}{-} \sum_{i=0}^{n-1} Q_i$$

and the sets V_n and $V_n^{(\infty)}$ defined by

$$V_n = \{z \mid \pi\Phi^n z \in W_n\}$$

$$V_n^{(\infty)} = \{z \mid \pi\Phi^n z \in W_n^{(\infty)}\} .$$

And we claim that V_n and $V_n^{(\infty)}$ are the sets of capturable points,

respectively, when $v(n)$ is known of the pursuer as step n, and when the whole future history $v(\cdot)$ is known.

The proof for V_n is rigorously the same as in Chapter 1; there is no need to repeat it. We prove the claim for $V_n^{(\infty)}$ since the situation is slightly different: we add steps together instead of letting the step grow up to the capture time.

We have

$$\pi z(n) = \pi\Phi^n z(0) - \sum_0^{n-1} u_k(n-1-k) + \sum_0^{n-1} v_k(n-1-k)$$

where

$$u_k \triangleq \pi\Phi^k u \qquad v_k = \pi\Phi^k v\ .$$

For simplicity of notation, let

$$\varphi_n = \sum_0^{n-1} u_k(n-1-k) \qquad \varphi_n \in \sum_0^{n-1} P_k$$

$$\psi_n = \sum_0^{n-1} v_k(n-1-k) \qquad \psi_n \in \sum_0^{n-1} Q_k\ .$$

For capture to be possible in n steps, it is necessary and sufficient that there exist a φ_n such that

$$\pi\Phi^n z(0) - \varphi_n + \psi_n \in \mathcal{C}\ ,$$

or equivalently, that

$$\pi\Phi^n z(0) + \psi_n \in \mathcal{C} + \sum_0^{n-1} P_k\ .$$

And, for this to be possible for every ψ_n, it is necessary and sufficient that

$$\pi\Phi^n z(0) + \sum_0^{n-1} Q_k \subset \mathcal{C} + \sum_0^{n-1} P_k$$

or equivalently that

$$\pi\Phi^n z(0) \in \left(\mathcal{C} + \sum_0^{n-1} P_k\right) \overset{*}{-} \sum_0^{n-1} Q_k ,$$

which proves the claim.

3.2 Capture with No Information on v

In the continuous case we had the possibility of letting ϵ go to zero. Here, if we want to have the information advantage of the pursuer vanish, the only thing we can do is assume that he has no information on v . Then, as we want capture to be possible whatever v is, it must be possible if the evader plays "as if he knew" the pursuer's control. Thus, we are actually led to the study of the majorant game, which was not needed in the continuous case.

By analogy with the previous constructions, we are looking for a set $\hat{W}_n$ such that

$$\pi\Phi^n z(0) \in \hat{W}_n \tag{3.2}$$

insures that it is always possible for the pursuer to obtain

$$\pi\Phi^{n-1} z(1) \in \hat{W}_{n-1} \tag{3.3}$$

and, in addition,

$$\hat{W} = \mathcal{C} .$$

Inclusion (3.3) reads

$$\pi\Phi^n z(0) - u_{n-1}(0) + v_{n-1}(0) \in \hat{W}_{n-1} \qquad \forall\, v_{n-1} \in Q_{n-1} .$$

Therefore

$$\pi\Phi^n z(0) - u_{n-1}(0) + Q_{n-1} \subset \hat{W}_{n-1} ;$$

equivalently

$$\pi\Phi^n z(0) - u_{n-1}(0) \in \hat{W}_{n-1} \overset{*}{-} Q_{n-1} .$$

The existence of such a u is equivalent to

$$\pi\Phi^n z(0) \in (\hat{W}_{n-1} \overset{*}{-} Q_{n-1}) + P_{n-1} \tag{3.4}$$

and as this must be a consequence of (3.2),

$$\hat{W}_n \subseteq (\hat{W}_{n-1} \overset{*}{-} Q_{n-1}) + P_{n-1} \, . \tag{3.4a}$$

More precisely, as (3.4) is a necessary and sufficient condition to insure that (3.3) is possible, we replace (3.4a) by the equality, and using it recursively together with $\hat{W}_o = \mathcal{C}$, define $\hat{W}_n$:

$$\hat{W}_n = \left(\left(\cdots\left(\left(\left(\mathcal{C} \overset{*}{-} Q_o\right) + P_o\right) \overset{*}{-} Q_1\right) + P_1\right)\cdots\right) \overset{*}{-} Q_{n-1}\right) + P_{n-1} \, .$$

<u>Remarks</u>

As could be expected, $\hat{W}_n$ does not exist unless, in particular, $\mathcal{C} \supseteq Q_o$. The relative size of the three sets W_n is easy to check:

$$\hat{W}_n \subseteq W_n \subseteq W_n^{(\infty)}$$

by straightforward application of the propositions of Section 1.6. They may also be used to establish that if $P_{i-1} \supseteq Q_i$, for every non-negative i , with $P_{-1} = \mathcal{C}$, and if $P_{i-1} \overset{*}{-} Q_i$ has complete sweeping, then

$$\hat{W}_n = W_n = W_n^{(\infty)} \, .$$

3.3 Concluding Remarks

We have not said anything about optimality so far.

In the case of $V_n^{(\infty)}$, later referred to as the strong controllability case, we have seen that $\pi\Phi^n z(0) \in V_n$ to insure $z(1) \in V_{n-1}$. This corresponds to the local theorems of Chapter Two. But we have the same "global" problem. We are not sure that the evader can simultaneously prevent the state from penetrating every V_i of smaller index.

One should therefore either find an equivalent of condition B, but this could be more difficult than in the continuous case, or redefine the family V_n in such a way that V_{n+1} includes all points z such that

the evader cannot prevent the state from drifting in one step into V_n simultaneously with the $n-1$ previous sets V_i .

We did not investigate this problem. We notice only that if $V_n \supset V_{n-1}$ for every n , then the problem does not arise, and the previous construction yields the optimal capture time.

3.4 System Theoretic Formulation

We turn now to the discrete system with unbounded controls. We must obviously reintroduce the matrices G and J through which they act. We also change our notations to more traditional ones.

We deal with the system

$$x(k+1) = F\,x(k) - G\,u(k) + J\,v(k) \tag{3.5}$$

where

$x \in X$ an n-dimensional vector space
F is an $n \times n$ constant matrix
$u \in U$ an m-dimensional vector space
$v \in V$ an m'-dimensional vector space
G and J are $n \times m$ and $n \times m'$ constant matrices.

A subspace M of X is given, and capture is defined as $x \in M$. We choose a complement L of M : $L \oplus M = X$ and we define π as the projection onto L parallel to M :

$$\pi x \in L \qquad\qquad x - \pi x \in M$$

and capture is equivalent to $\pi x = 0$.

We introduce, in addition, the notations

$$P_k = \pi \text{ range } (F^k G) \qquad Q_k = \pi \text{ range } (F^k J) \text{ .}$$

P_k and Q_k are vector subspaces of L .

We can still define a geometric subtraction: given two subspaces A and B ,

$$A \stackrel{*}{-} B = D = \{x \mid x + B \subset A\} \text{ .}$$

But this operation is now particularly simple. Two cases arise:

$A \supseteq B \qquad\qquad A \overset{*}{-} B = A$ with complete sweeping

$A \not\supseteq B \qquad\qquad A \overset{*}{-} B = \emptyset$ or, equivalently, does not exist.

And finally, from the Cayley-Hamilton Theorem,

$$\sum_0^p P_i = \text{range } \{\pi G, \pi FG, \ldots, \pi F^p G\} \subseteq \sum_0^{n-1} P_i \qquad \forall\, p$$

and similarly for the Q_i's , so that we can stop all our constructions at n steps.

3.5 Strong Controllability, Capturability and Ideal Capturability

We apply the same technique as previously, with vector subspaces. We again have the three main information structures:

α) Strong Controllability. The control $v(k)$ is known for the whole future. If the state can be brought to the origin, we shall say, following Kalman [19], that it is strongly controllable modulo M .

We have

$$W_k^{(\infty)} = \sum_0^{k-1} P_i \overset{*}{-} \sum_0^{k-1} Q_i \ .$$

Therefore

$$W_k^{(\infty)} = \sum_0^{k-1} P_i \quad \text{if} \quad \sum_0^{k-1} P_i \supseteq \sum_0^{k-1} Q_i$$

$$W_k^{(\infty)} = \emptyset \quad \text{if} \quad \sum_0^{k-1} P_i \not\supseteq \sum_0^{k-1} Q_i \ .$$

The condition for the existence of $W_k^{(\infty)}$ can be written as an explicit condition on the coefficients of the matrices involved:

$$\text{rank } [\pi G \,|\, \pi FG \,|\, \ldots \pi F^{k-1} G] = \text{rank } [\pi G \,|\, \pi J \,|\, \pi FG \,|\, \pi FJ \,|\, \ldots \, \pi F^{k-1} G \,|\, \pi F^{k-1} J] \ .$$

Notice that $W_k^{(\infty)}$ may exist (be non-empty) while a set of lower order would not. The states of the corresponding $V_k^{(\infty)}$ are still strongly controllable, since, as we saw in Section 3.1, the argument for that case does not proceed by induction.

We find that for every k, either all states controllable with u alone in exactly k steps are strongly controllable in exactly k steps, or none is. In the latter case, however, some states may still be strongly controllable in less than k steps. But the pursuer will not, then, be able to keep the state in M until time k, or to have it reach M at time k only.

β) Capturability. The value of v at the present step is known. If a state can be brought into M with that information, we shall say that it is capturable modulo M.

We have

$$W_k = \left(\left(\dots\left(\left(\left(P_o \stackrel{*}{-} Q_o\right) + P_1\right) \stackrel{*}{-} Q_1\right) + P_2 \ \dots\right) + P_{k-1}\right) \stackrel{*}{-} Q_{k-1} \ .$$

Therefore one of the two following possibilities must arise:

- All geometric differences non-empty $\qquad W_k = \sum_0^{k-1} P_i$
- Otherwise $\qquad W_k = \emptyset$.

The condition for W_k to be non-empty is

$$P_o \supseteq Q_o \qquad (P_o+P_1) \supseteq Q_1 \qquad \dots \qquad \sum_0^{k-1} P_i \supseteq Q_{k-1}$$

which can be transcribed in terms of the matrices, considering π as the matrix corresponding to the projection operation

$$\text{rank } [\pi G \mid \pi J] = \text{rank } [\pi G] \qquad \text{rank } [\pi G \mid \pi F G \mid \pi F J]$$
$$= \text{rank } [\pi G \mid \pi F G \text{ , etc.}$$

It suffices that $P_q \supseteq Q_q$ or $\text{rank } [\pi F^q G \mid \pi F^q J] = \text{rank } [\pi F^q G] \quad \forall q$; it suffices also that $P \supseteq Q$ or $\text{rank } [G \mid J] = \text{rank } [G]$.

These three conditions are increasingly restrictive; the first one only is necessary. Because π and F generally do not commute, $P_o \supseteq Q_o$ does not imply the second condition. Notice that the existence of W_k implies, as it should, the existence of $W_k^{(\infty)}$. The interesting fact is that they are equal.

The conclusion about the capturability modulo π in exactly k steps is the same as in the case of the strong controllability. Only the conditions under which the controllable states are capturable are more stringent.

γ) Ideal Capturability. No information is available on v. By analogy with the concept of ideal observability, we shall say of a state that can be brought into M in that case that it is ideally capturable modulo M. We have

$$\hat{W}_k = \left(\left(\cdots\left(\left(P_o \stackrel{*}{-} Q_1\right) + P_1\right) \stackrel{*}{-} Q_2 \cdots\right) \stackrel{*}{-} Q_{k-1}\right) + P_{k-1} .$$

Therefore, two possibilities again arise:

- All geometric differences are non-empty $\qquad \hat{W}_k = \sum_0^{k-1} P_i$
- Otherwise $\qquad \hat{W}_k = \emptyset$.

The condition for $\hat{W}_k$ to exist is

$$Q_o = \{0\} \qquad P_o \supseteq Q_1 \qquad P_o + P_1 \supseteq Q_2 \cdots \sum_0^{k-1} P_i \supseteq Q_{k-1}$$

and this condition can be written in terms of the matrices:

$$\pi J = 0 \qquad \text{rank}\,[\pi G \mid \pi F J] = \text{rank}\,[\pi G] \qquad \text{rank}\,[\pi G \mid \pi F G \mid \pi F^q J] = \text{rank}\,[\pi G \mid \pi F G] ,$$

etc.

Again, it suffices that $P_{q-1} \supseteq Q_q$, $\text{rank}\,[\pi F^{q-1} G \mid \pi F^q J] = \text{rank}\,[\pi F^{q-1} G]$ $\forall q$ but the first condition only is necessary.

The situation is similar to what it was in the two previous cases, with even more restrictive conditions that clearly imply the existence of the two other sets, and then they all are equal.

3.6 Discussion, Optimality

We find that in the case of unbounded controls, unlike in the other case, a change in the information structure does not change the nature of the sets of controllable states. It only changes the condition under which these sets have the desired property.

In other words, if some states are, say, ideally capturable in k steps, then all states controllable in k steps are strongly controllable, capturable and ideally capturable. What can be changed by the information structure is the subspace M, modulo which the system has the discussed properties. In particular, changing the information may allow us to bring more coordinates of the state to zero. This is not in contradiction with our previous statement which holds for a fixed subspace M.

As far as the optimality of the capture time is concerned, we have, of course, the same "step by step" optimality as in Section 3.3. But the "global" problem is now much simpler.

We consider the relation

$$x(1) = F\,x(0) - G\,u(0) + J\,v(0) \in V_k ,$$

and we know that the pursuer can achieve this if, letting $P = \text{range } G$,

$$F\,x(0) + J\,v(0) \in V_k + P . \tag{3.6}$$

As V_k+P is a vector space, (3.6) is equivalent to a set of linear equations on v. If $x(0)$ does not belong to V_{k+1}, then, by definition, it is not verified identically. Then, the set of all v's that verify it is an affine set in V, possibly empty. The union of a finite number of such sets cannot be the whole space V.

Thus, if $T(x) = p : x \in V_p$ and $x \notin V_k \;\; \forall\, k < p$, then there are v's for which none of the relations (3.6), with every k smaller than p, is verified.

This solves the problem by showing that p is indeed optimal.

3.7 Invariant Capture Space

We are going to investigate the case where the subspace M is invariant under F. A reason for doing so is that it corresponds to a natural problem in the frame of modern algebraic system theory. M is then a submodule of the module structure induced on X by polynomials in F.

The results take a simple form, and we are able to generalize to the multiple input case a result proved by Kalman [19] in the single input case.

i) The Strong Controllability Theorem. We first prove two simple lemmas.

Lemma 1: If a state is controllable modulo M in p steps, it is also controllable in $p+q$ steps, $q > 0$. This is an immediate consequence of the invariance of M. Translated in our notations, this implies:

$$\pi F^p x \in \sum_0^{p-1} P_i \Rightarrow \pi F^{p+q} x \in \sum_0^{p+q-1} P_i \qquad \forall\, q \geq 0 .$$

Lemma 2: If $W_p^{(\infty)}$ is non-empty, then $W_{p+q}^{(\infty)}$ is not empty either, for every $q \geq 0$.

Proof: Assume

$$\sum_0^{p-1} P_i \supseteq \sum_0^{p-1} Q_i .$$

This means that for every sequence $v_o, v_1, \ldots v_{p-1}$, there exists a corresponding sequence $u_o, u_1, \ldots u_{p-1}$ such that

$$\sum_0^{p-1} \pi F^k G u_k = \sum_0^{p-1} \pi F^k J v_k ,$$

or equivalently

$$\sum_0^{p-1} F^k G u_k - \sum_0^{p-1} F^k J v_k \in M .$$

Using the invariance of M under F, we multiply by F^q

$$\sum_0^{p-1} F^{k+q}Gu_k - \sum_0^{p-1} F^{k+q}Jv_k \in M ,$$

and as this is possible for every sequence v_k, it is equivalent to

$$\sum_q^{p+q-1} P_i \supseteq \sum_q^{p+q-1} Q_i ,$$

which together with the relation we started from gives

$$\sum_0^{p+q-1} P_i \supseteq \sum_0^{p+q-1} Q_i \quad \text{or} \quad W_{p+q}^{(\infty)} = \sum_0^{p+q-1} P_i \neq \emptyset .$$

We can now prove the following theorem:

Theorem: When M is invariant under F, then

- Either all the states controllable with v alone (modulo M), are controllable with u, and then all the states controllable with u are strongly controllable;
- Or no state is strongly controllable.

Proof: Assume that some states are strongly controllable. Then there exists a non-empty $W_k^{(\infty)}$, and thus, by Lemma 2, $W_n^{(\infty)}$ is non-empty.

$$\sum_0^{n-1} P_i \supseteq \sum_0^{n-1} Q_i . \tag{3.7}$$

By Lemma 1, all states controllable with u are given by

$$\pi F^n x \in \sum_0^{n-1} P_i$$

and all states controllable by v similarly with the Q_i's. Thus, if (3.7) is verified, all states controllable with v are controllable with u, and all states controllable with u verify

$$\pi F^n x \in W_n^{(\infty)}$$

and thus are strongly controllable.

If (3.7) is not verified, no state is strongly controllable, since it is verified as soon as some are. This ends the proof.

ii) Absolute Concepts. We want to investigate under what condition, once the pursuer has brought the state in M, he will be able to hold it in M. When he is able to do so, we shall say that the system is absolutely strongly controllable, capturable or ideally capturable. Conditions for this to happen in the case where M is not invariant can be given, but they are not very interesting. Here, with M invariant, the situation is very simple.

Let us first make a few remarks about this question in the case of the strong controllability. Let ℓ be the smallest integer for which $W_\ell^{(\infty)}$ is not empty. Then $W_\ell^{(\infty)}$ contains the origin. Thus, with the invariance of M, if the state has been captured at time p, we have

$$x(p) \in M \qquad F^\ell x(p) \in M \qquad \pi F^\ell x(p) = 0 \in W_\ell^{(\infty)}$$

and the pursuer is able to have the state return to M every ℓ instants of time. However, if he wants the state to belong to M at a given instant m larger than $p+\ell$, $m = p+q$, he can always achieve this since

$$F^q x(p) \in M \qquad\qquad \pi F^q x(p) = 0 \in W_q^{(\infty)}$$

and we know that $W_q^{(\infty)}$ does exist.

If we want the system to be absolutely strongly controllable, then $W_1^{(\infty)}$ must exist:

$$P_o \supseteq Q_o \ .$$

This insures $P_k \supseteq Q_k$ due to the invariance of M, as is easily checked: if for every v there is a u such that

$$\pi Gu = \pi Jv \qquad\qquad Gu - Jv \in M\ ,$$

we can multiply both sides of the second inclusion by F^k, which gives the result. But this implies capturability. We therefore have the following result:

Theorem: When M is invariant under F, the concepts of absolute strong controllability, capturability and absolute capturability are equivalent.

The problem of ideal capturability is of no interest when M is invariant, since ideal capturability requires that the range of J be in M, and then the evader would have no control on πx.

This finishes our discussion of the discrete problem.

4. THE ISOTROPIC ROCKET GAME AS AN EXAMPLE

In this short chapter, we present a special pursuit-evasion game: the Isotropic Rocket Game (I.R.G.). Our aim is to discuss its formulation and to apply to it the results of the previous theory.

4.1 Description of the I.R.G.

The Isotropic Rocket Game was proposed by Rufus Isaacs in [17] and [18]. In these references, Isaacs gave an analysis which, although farther than ours from being complete, brought out several new and interesting features. This analysis covers most of what we present in Sections 5.2, 5.3 and 6.2. We shall often refer to this work. We have tried to stay as close as possible to the notations of [18]. However, it was not always possible to keep exactly the same notation, partly because this game appears at two different places in the book, with different notations. A correspondence between ours and those of these two discussions is given in Appendix C.

In this game, the dynamical possibilities of the two players are as follows:

P. The pursuing object is to be thought of as a rocket able to direct its thrust in any direction, whence the name of the game. It has a bounded thrust-to-mass ratio, that is, an acceleration the magnitude of which cannot exceed a fixed value F. Within this restriction, this acceleration can be changed instantly and is the pursuer's control.

E. The pursued object is a maneuverable target, with bounded velocity. The maximum possible magnitude of this velocity is w. Within this restriction, it can be changed instantly and is the evader's control.

Notice that neither of these two descriptions is very realistic. A rocket is not steered by instantly changing the direction of its thrust, and we allow the target, an aircraft or an incoming missile, for instance, infinite accelerations. However, simplified as it is, this model will still give meaningful non-trivial results about the chase. In addition,

it will yield new concepts for a general theory of differential games.

Capture is obtained when the relative distance of the two players falls below a fixed radius of capture ℓ . This can represent two players of finite radii ℓ_1 and ℓ_2 with $\ell_1+\ell_2 = \ell$, or a pursuer with lethal radius ℓ pursuing a point-like target. We shall think of it in this second and more realistic way, knowing that the analysis is equally valid for the other case. Whether the capture set C must be regarded as an open or a closed sphere is not important at the modelling stage. Depending on the techniques used, the question will be answered in the way that best fits the mathematical formulation.

4.2 Dimension of the Geometrical Space

The chase occurs in the three-dimensional physical space, but we neglect gravity. To recover Isaacs' two-dimensional formulation, we immediately state the following fact:

Proposition: An optimal chase occurs in a fixed plane.

Proof: Let $\vec{r}$ be the vector from P (center of the capture sphere) to E , and $\vec{v}$ be P's velocity. Consider the plane Π defined at each instant by P , $\vec{r}$ and $\vec{v}$, which thus contains E . Take a moving rectangular coordinate system (x,y,z) with its origin at the point P , and such that the x- and y-axes are in Π : for instance, the y-axis aligned with $\vec{v}$. In these axes, the relative coordinates of E are

$$\vec{r} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

and P's velocity is

$$\vec{v} = \begin{pmatrix} 0 \\ v \\ 0 \end{pmatrix} .$$

This coordinate system has an angular velocity $\vec{\omega}$ with respect to the inertial space:

$$\vec{\omega} = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} .$$

Consider a new coordinate system (X,Y,z), still rectangular, with the same origin and the same z-axis, but having, with respect to the previous one, an angular velocity $-\omega_z$. This new system has, with respect to the inertial space, an angular velocity $\vec{\Omega}$:

$$\vec{\Omega} = \begin{pmatrix} \omega_X \\ \omega_Y \\ 0 \end{pmatrix}$$

and $\vec{r}$ and $\vec{v}$ have for components

$$\vec{r} = \begin{pmatrix} X \\ Y \\ 0 \end{pmatrix} \qquad \vec{v} = \begin{pmatrix} U \\ V \\ 0 \end{pmatrix} .$$

Let $\frac{\partial}{\partial t}$ denote the time derivative with respect to the axes (X,Y,z) and $\frac{d}{dt}$ the time derivative with respect to the inertial space. We decompose every vector on the (X,Y,z) axes. For any vector $\vec{a}(t)$, we have by definition

$$a(t) = \begin{pmatrix} \alpha(t) \\ \beta(t) \\ \gamma(t) \end{pmatrix} \qquad \frac{\partial\vec{a}}{\partial t} = \begin{pmatrix} \dot{\alpha}(t) \\ \dot{\beta}(t) \\ \dot{\gamma}(t) \end{pmatrix}$$

and, by the classical laws of kinematics,

$$\frac{\partial\vec{a}}{\partial t} = \frac{d\vec{a}}{dt} - \vec{\Omega} \times \vec{a} .$$

Applying this to $\vec{r}$ and noticing that $\frac{d\vec{r}}{dt} = \vec{w} - \vec{v}$, we find

$$\dot{X} = w_X - U$$

$$\dot{Y} = w_Y - V$$

$$0 = w_z + X\omega_Y - Y\omega_X$$

and similarly with $\frac{d\vec{v}}{dt} = \vec{F}$, which gives

$$\dot{U} = F_X$$

$$\dot{V} = F_Y$$

$$0 = F_z + U\omega_Y - V\omega_X \ .$$

If we notice that capture is defined by $X^2+Y^2 \leq \ell^2$, we see that all the information is contained in the four-dimensional game in X,Y,U,V . For this game, the controls are the projections (w_X,w_Y) and (F_X,F_Y) of $\vec{w}$ and $\vec{F}$ on Π .

Moreover, the dynamical equations of that game are linear, of type (1.1). We can apply the optimality principle, derived in Chapter Two, and we find that the optimal strategies must verify

$$w_X^2 + w_Y^2 = w^2 \qquad\qquad F_X^2 + F_Y^2 = F^2$$

and therefore

$$w_z = 0 \qquad\qquad F_z = 0 \ .$$

Placing this in the z equations of our two differential systems yields:

$$\omega_X = \omega_Y = 0 \ .$$

(When $\vec{r}$ and $\vec{v}$ are aligned, we can choose this solution.)

Therefore, the coordinate system (X,Y,z) has a fixed direction in inertial space. As a consequence, the plane Π , in which the chase occurs, can be considered as fixed in space.

This proves the proposition.

4.3 Representations

Two main representations of our dynamics will be used.

i) 4-D Representations. The first one is four-dimensional. The origin of the coordinate system is at the center of the pursuer's circle of capture. The orientation of the axes is fixed in inertial space. The state variables are the relative coordinates X,Y of the evader, and the components U,V of the pursuer's velocity.

The equations of motion are

$$\begin{aligned} \dot{X} &= -U + w_X \\ \dot{Y} &= -V + w_Y \\ \dot{U} &= F_X \\ \dot{V} &= F_Y \end{aligned} \tag{4.1}$$

with

$$w_X^2 + w_Y^2 \leq w^2 \qquad\qquad F_X^2 + F_Y^2 \leq F^2 \ .$$

This form is linear, and the previous theory will apply to it directly. Notice a vectorial formulation of it, with vectors of the geometrical space:

$$\vec{r} = \begin{pmatrix} X \\ Y \end{pmatrix} \qquad\qquad \vec{v} = \begin{pmatrix} U \\ V \end{pmatrix}$$

$$\begin{aligned} \dot{\vec{r}} &= -\vec{v} + w\hat{\alpha} \\ \dot{\vec{v}} &= F\hat{\beta} \end{aligned} \tag{4.2}$$

where $\hat{\alpha}$ and $\hat{\beta}$ are unit vectors, the direction of which are the controls. Here it has already been assumed that the players choose their controls on the boundary of the control sets.

ii) <u>3-D Representations</u>. It is possible to find a three-dimensional representation: its equations are much more complicated than (4.1), and non-linear, but it will be desirable in the subsequent theory to use the lowest dimensional representation.

The game is obviously insensitive to absolute orientation in the plane. We can take advantage of this by choosing the y-axis, for instance, parallel to the pursuer's velocity. Then this velocity is represented by a single variable, its magnitude.

Again, we assume that both players choose controls of maximum magnitude, so that we can represent their controls by a single parameter for each. Following Isaacs, we choose to give the directions of these controls by their angle, measured clockwise from the y-axis: φ for the

pursuer and ψ for the evader.

The equations of motion are easy to derive. They are (see Appendix A):

$$\dot{x} = -\frac{Fy}{v}\sin\varphi + w\sin\psi$$

$$\dot{y} = \frac{Fx}{v}\sin\varphi + w\cos\psi - v \tag{4.3}$$

$$\dot{v} = F\cos\varphi\ .$$

These coordinates are related to the previous ones through the formulas

$$v = \sqrt{U^2+V^2}$$

$$x = \frac{1}{v}(-UY + VX) \tag{4.4}$$

$$y = \frac{1}{v}(UX + VY)\ .$$

In this system, the capture set is a cylinder of revolution around the v-axis. As a consequence, we shall also use the cylindrical form of the same coordinates:

$$x = r\sin\theta$$

$$y = r\cos\theta$$

and the equations of motion now are

$$\dot{r} = w\cos(\psi-\theta) - v\cos\theta$$

$$\dot{\theta} = -\frac{F}{v}\sin\varphi + \frac{w}{r}\sin(\psi-\theta) + \frac{v}{r}\sin\theta \tag{4.5}$$

$$\dot{v} = F\cos\varphi\ .$$

The various coordinate systems are depicted in Fig. 2.

iii) Parameter. The unit of length can be chosen arbitrarily, so as to assign any desired numerical value to ℓ . This being done, the unit of time can still be chosen so as to assign any desired value to w . Then the game is completely defined by a single numerical value for F .

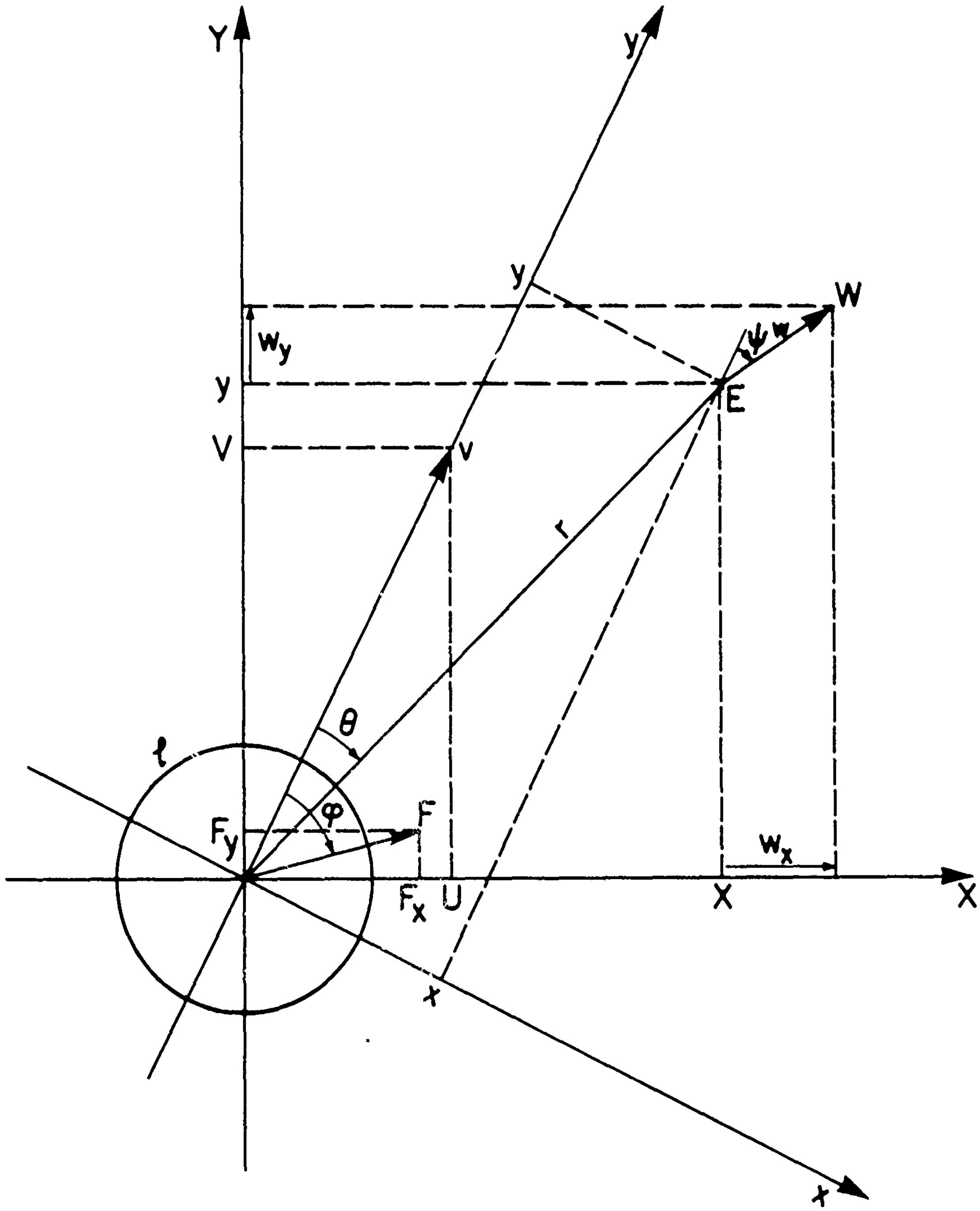

FIGURE 2. The Coordinate Systems

In our analysis, we have chosen not to nondimensionalize, in order to let the nature and meaning of intermediary quantities and relations be more apparent. But the previous remark shows that a single non-dimensional parameter is needed to characterize the game. We shall use

$$p = \frac{w^2}{2F\ell} .$$

The factor two in the denominator has been put there for reasons of convenience that will appear later.

4.4 Results From the Previous Theory

i) Formulation. We use the linear representation, and we define:

$$z = \begin{pmatrix} X \\ Y \\ -U \\ -V \end{pmatrix} \qquad u = \begin{pmatrix} 0 \\ 0 \\ F_X \\ F_Y \end{pmatrix} \qquad v = \begin{pmatrix} w_X \\ w_Y \\ 0 \\ 0 \end{pmatrix}$$

And the matrix C is then

$$C = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad e^{\tau C} = \Phi(\tau) = \begin{pmatrix} 1 & 0 & \tau & 0 \\ 0 & 1 & 0 & \tau \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

P and Q are disks in their respective subspaces. The geometrical subspace is the subspace of the first two coordinates, in which capture is defined by

$$\mathcal{C} = \{z \mid X^2 + Y^2 \leq \ell^2\}$$

where, to comply with the formulation of our theory, $\mathcal{C}$ has been chosen as a closed set.

The operator $\pi\Phi(\tau)$ is given by the matrix

$$\pi\Phi(\tau) = \begin{pmatrix} 1 & 0 & \tau & 0 \\ 0 & 1 & 0 & \tau \end{pmatrix}$$

so that P_τ and Q_τ are circles centered at the origin and of respective radii τF and w. Notice that whatever the relative value of the parameters, for small enough τ, $P_\tau \subset Q_\tau$.

It is shown in Pontryagin [24] that this implies that, if the evader knows the present control of the pursuer, capture is impossible if it is defined as point coincidence: $\ell = 0$. We shall reach this same conclusion without the assumption that the pursuer's control is known by the evader (see Chapter 5). In fact, we are in a case where the sets W_τ are disks, and have a single normal at each point of their boundary. Therefore, the concluding remark of Section 2.5 holds; (2.7) and (2.7a) define unambiguously the controls u^* and v^* , independently of each other.

ii) Estimating Function. The sets $\mathcal{C}$, P_τ and Q_τ are all disks. After Pontryagin, we notice that in this case the operations of sum, geometric difference and integral of sets reduce to sum, difference and integral of radii. The alternating integral

$$W_\tau = \int_{\mathcal{C},0}^{\tau} [P_r \overset{*}{-} Q_r]dr$$

is the disk centered at the origin, and of radius $Q(\tau)$ given by

$$Q(\tau) = \ell + \int_0^\tau (rF-w)dr = F\frac{\tau^2}{2} - w\tau + \ell \qquad (4.6)$$

and W_{τ_0} exists as long as $Q(\tau)$ is non-negative for every τ smaller than τ_0 . ζ is given by

$$\zeta(\tau) = \pi\Phi(\tau)z = \begin{pmatrix} X-\tau U \\ Y-\tau V \end{pmatrix}$$

so that $T(z) = \tau_0$ is the smallest positive root of the equation:

$$(X-\tau U)^2 + (Y-\tau V)^2 = Q(\tau)^2 \ . \qquad (4.7)$$

We can use the formulation (4.2) to express (4.7) in a different form

$$\vec{\zeta} = \vec{r} - \tau\vec{V}$$

so that (4.7) becomes

$$\|\vec{r} - \tau\vec{V}\| = Q(\tau) \ . \qquad (4.7a)$$

We notice that $Q(\tau)$ is quadratic in τ , when $\|\zeta(\tau)\|$ is linear. Thus

equation (4.7a) always has a solution if $\|\zeta(0)\| \geq Q(0)$, namely $\|\vec{r}\| \geq \ell$, which is always verified for the starting point.

This means that if $Q(\tau)$ does not vanish, capture will always occur in finite time. $Q(\tau)$ never vanishes if its determinant is negative:

$$w^2 - 2F\ell < 0 \quad \text{equivalently} \quad p < 1 .$$

We therefore have the following important result:

Proposition: For p smaller than one, capture occurs from all initial conditions.

iii) Barrier. We have a somewhat simpler way of using equation (4.7a). Instead of translating r by $-v\tau$, we prefer to translate W_τ by $+v\tau$ and directly check whether the point considered belongs to this capture set.

Drawing these sets for a given v , we obtain Fig. 3, which is the same as Fig. 5.5.4, p. 114, in [18], although obtained by completely different means.

The most prominent feature of this figure is the existence of an envelope. It is a line of non-regular points of first kind, or barrier. Its equation is easy to establish. We use the axes of the three-dimensional representation. Then the circles verify the equation:

$$x^2 + (y-v\tau)^2 = \left(\frac{1}{2}F\tau^2 - w\tau + \ell\right)^2$$

and their envelope is given parametrically by

$$\begin{aligned} y - v\tau &= \frac{w-F\tau}{v} Q(\tau) \\ x &= \pm \frac{\sqrt{-F^2\tau^2 + 2wF\tau + v^2 - w^2}}{v} Q(\tau) . \end{aligned} \tag{4.8}$$

The double sign in x accounts for the two symmetric parts of the envelope. Here, the parameter τ is the estimated time to go just inside of the discontinuity, and on the barrier itself, which verifies the conditions of the last theorem of Section 2.11. (But the trajectories of

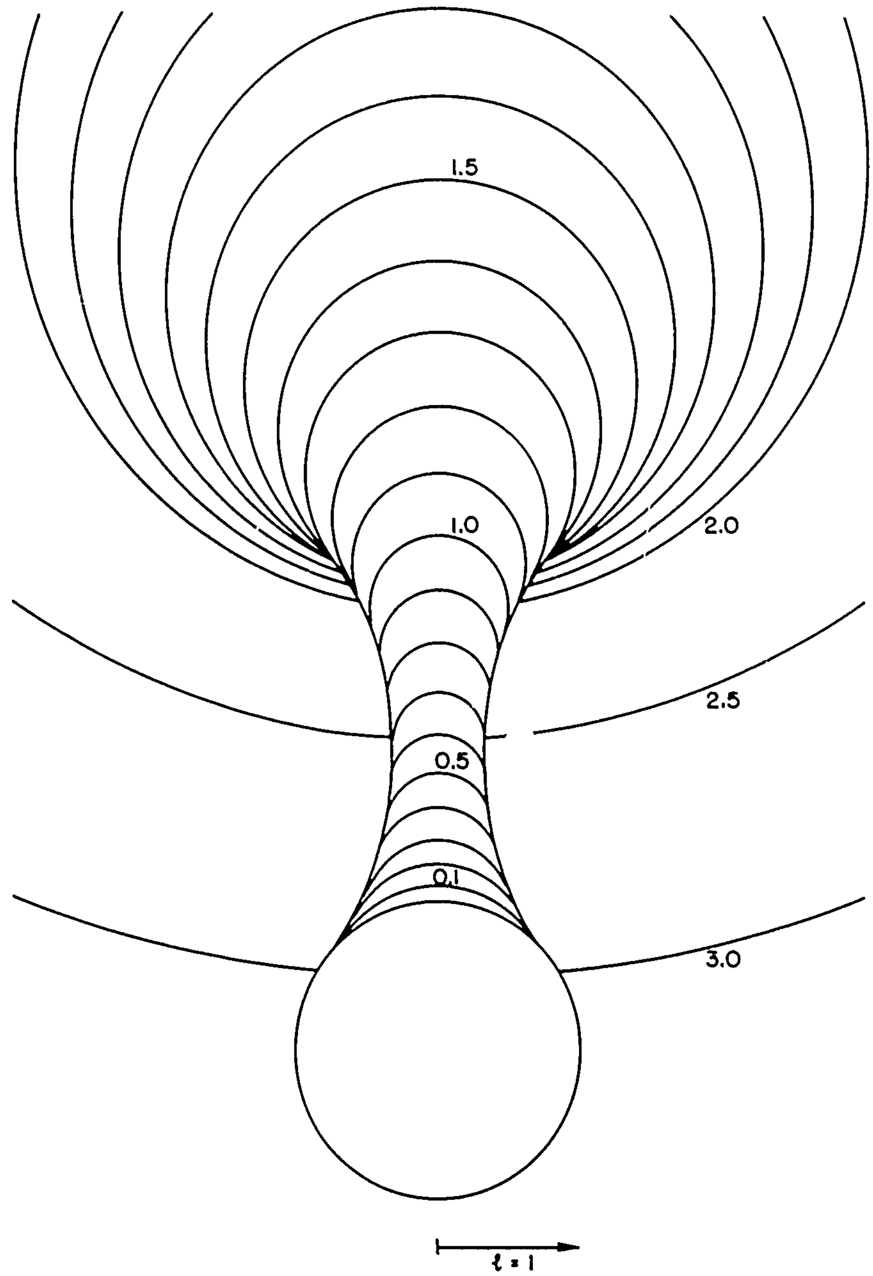

FIGURE 3. The Translated W_T's for F-3, $w = 2$, $v = 1.5 \times w = 3$

the barrier do not have the shape of the envelope (4.8), since they have a varying v .)

When $p > 1$, $Q(\tau)$ vanishes for some τ_1 , this envelope defines a curvilinear triangle of increasing size as v is increased. The locus of its vertex on the y-axis is a straight line at:

$$x = 0$$

$$y = v\tau_1 \ .$$

We shall later refer to this line as the "crest." We are insured that, inside this region, capture will always occur.

4.5 Conclusion

The technique developed in the first part has given us a positive answer to the problem of completion: for $p < 1$ capture is always possible. Moreover, if we can check that the trajectories do not cross the barrier or penetrate the capture circle, we have the optimal time of capture and the optimal strategies. This will be found to be the case for a large region of the state space.

However, it will be seen that some of these trajectories would in fact penetrate $\mathcal{C}$. Consequently, we do not have the optimal strategies for the region these trajectories come from. Neither can we assert that for $p > 1$ evasion occurs from outside the region of finite $T(z)$.

This problem will be investigated in the next chapter by trying to construct directly all the barriers. It will be seen that escape is probably not possible unless p is larger than some p_o larger than one.

5. THE GAME OF KIND

In this chapter, we generalize the concept of barrier and use it to investigate the game of kind, the outcome of which is qualitative: capture or escape.

5.1 Semi-Permeable Surfaces

i) Analytical Description. In [18], Isaacs introduces the concept of semi-permeable surfaces. Let the dynamics of a game be (we use φ and ψ for the controls):

$$\dot{z} = f(z,\varphi,\psi) .$$

Let S be a surface and ν its normal. Suppose it is such that

$$\min_{\varphi} \max_{\psi} \langle \nu, f(z,\varphi,\psi) \rangle = 0 . \tag{5.1}$$

We shall always assume that f is "separated," that is, of the form

$$f(z,\varphi,\psi) = h(z,\psi) - g(z,\varphi)$$

so that

$$\min_{\varphi} \max_{\psi} \langle \nu, f(z,\varphi,\psi) \rangle = \max_{\psi} \langle \nu, h(z,\psi) \rangle - \max_{\varphi} \langle \nu, g(z,\varphi) \rangle .$$

Equation (5.1) has an obvious geometrical meaning: it states that player E cannot force the state to cross S in the direction of ν, when player P cannot force it to cross S in the other direction. Thus, if they both try to do so, the ensuing motion will be in S. S is called a semi-permeable surface. Its analogy with the barrier of Chapter Two is obvious, particularly in view of the last theorem of that chapter.

In particular, if such a surface defines a closed region containing the capture set, with ν pointing outside, the evader can make sure he will never penetrate this region if he starts from outside, and thus never be captured.

Even if this region is open, such a surface can still represent a discontinuity of the capture time if capture is only possible on one side of it.

The way to construct such surfaces is discussed in [18]; we shall indicate it only briefly.

Given a line of initial conditions, one first determines at each point of it a vector ν normal to this line and verifying (5.1). The same relation also determines φ and ψ, uniquely if the "vectorgrams" P and Q are strictly convex, and thus $\dot{z}$.

The differential equations for the normal ν along a trajectory are well known to be the adjoint equations. See, for instance, [3]. δz being a vector tangent to the surface, it verifies

$$\delta\dot{z} = \left(\frac{\partial f}{\partial z}\right)\delta z$$

and taking

$$\dot{\nu} = -\left(\frac{\partial f}{\partial z}\right)^{*}\nu$$

where the star denotes the adjoint operator, we have, defining q as

$$q = \langle \nu, \delta z\rangle$$

$$\dot{q} = \langle \nu, \frac{\partial f}{\partial z}\delta z\rangle + \langle -\left(\frac{\partial f}{\partial z}\right)^{*}\nu, \delta z\rangle = 0$$

so that if q is zero at some time, it is at every time, and we check that ν stays normal to the surface.

ii) Geometrical Description. Notice that (5.1) provides one relation only between z and ν, so that at each point of the state space, there usually exists a cone (hypercone) of "semi-permeable ν's ," and a corresponding cone of "semi-permeable directions" f. We propose a simple geometric construction of these two cones.

For a given point z, let $P = g(z,\varphi)$, $\varphi \in \Phi$ the set of allowable φ's, and let $Q = h(z,\psi)$, $\psi \in \Psi$ the set of allowable ψ's. Notice that as compared to our earlier definitions, the terms independent of the controls in f, Cz, for instance, have been arbitrarily cast into one of the functions h or g, thus translating P or Q by the same amount.

Let a bitangent plane be a plane (hyperplane) Π such that

1) Π contains one point at least of each of the sets P and Q ;

2) P and Q are both entirely contained in the same half space defined by Π .

Let ν be normal to Π , opposite to the half space containing P and Q .

<u>Proposition:</u> ν is a semi-permeable normal. The corresponding semi-permeable directions are the vectors joining any point of $\Pi \cap P$ to any point of $\Pi \cap Q$.

<u>Proof:</u> Let

$$g(z,\varphi^*) \in \Pi \cap P \qquad h(z,\psi^*) \in \Pi \cap Q \ .$$

Because of property two

$$\langle \nu, g(z,\varphi) \rangle \leq \langle \nu, g(z,\varphi^*) \rangle \qquad \forall\ \varphi \in \Phi$$

$$\langle \nu, h(z,\psi) \rangle \leq \langle \nu, h(z,\psi^*) \rangle \qquad \forall\ \psi \in \Psi \ .$$

Thus

$$\langle \nu, f(z,\varphi^*,\psi^*) \rangle = \min_{\varphi} \max_{\psi} \langle \nu, f(z,\varphi,\psi) \rangle \ .$$

Since $g(z,\varphi^*)$ and $h(z,\psi^*)$ both belong to Π , their difference is parallel to Π , hence normal to ν . Therefore, relation (5.1) is verified and the proposition proved (see Fig. 3).

iii) <u>The I.R.G.</u> In our case, with f given by (4.3), we can represent the vectorgram as follows (see Fig. 4).

In an $(\dot{x},\dot{y},\dot{v})$ space, visualized with its axes parallel to the (x,y,v) axes, Q is a circle of radius w centered at the origin and lying in the $\dot{x}\dot{y}$ plane; P is an ellipse centered at a point $(0,v,0)$ with one principal semi-axis of length F parallel to the $\dot{v}$ axis, and the other one, of length $\frac{Fr}{v}$ in the $\dot{x}\dot{y}$ plane normal to $\vec{r}$.

In the case drawn in Fig. 4, there are two separate cones of ν's given by our construction, one "above" the plane of P and one "under." Correspondingly, there are two cones of semi-permeable directions.

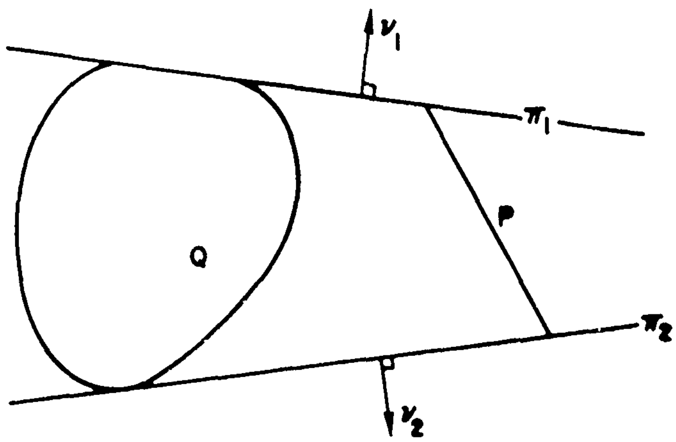

FIGURE 4. The Cone of Semipermeable Directions

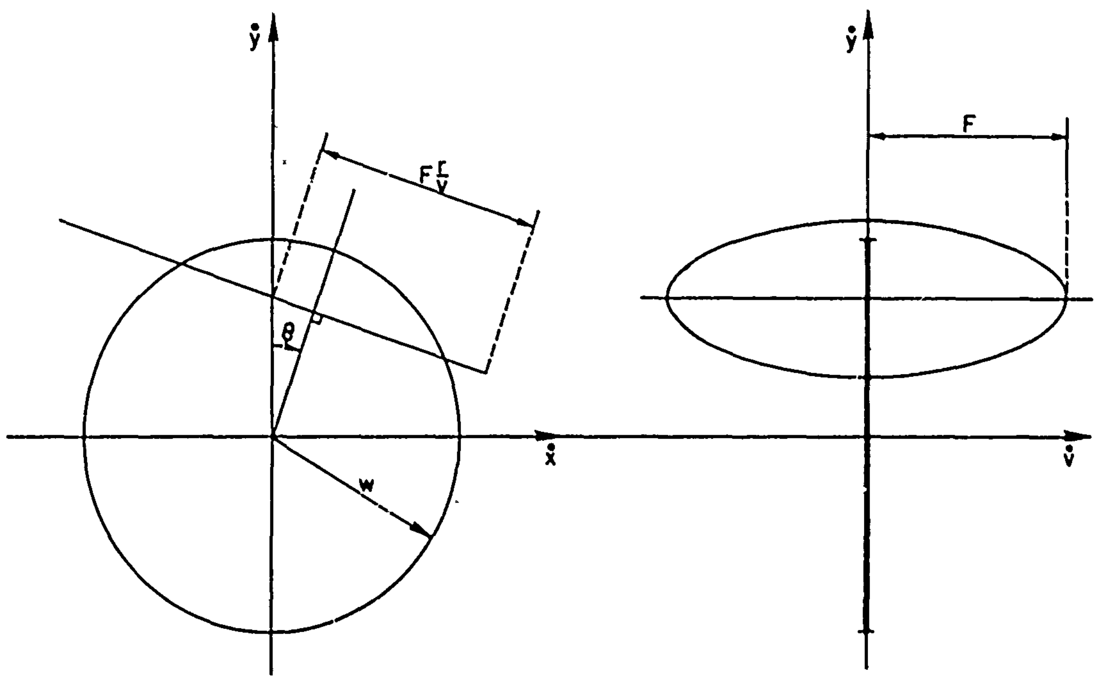

FIGURE 4a. The IRG Vectogram

If one extremity of the axis of P lying in the $\dot{x}\dot{y}$ plane is inside Q, then there is a single continuous family of ν's, and of semi-permeable directions. By elementary geometry, it is easy to see that we are in the first case for

$$Q_1 = \frac{F^2 r^2}{v^2} + v^2 - 2Fx - w^2 > 0 \ .$$

If $Q_1 \leq 0$, the two cones merge together to yield the second case. This family splits again into two separate cones, on each side of the $\dot{x}\dot{y}$ plane, when

$$Q_2 = \frac{F^2 r^2}{v^2} + v^2 + 2Fx - w^2 < 0 \ .$$

Notice that for positive x (the game being symmetric with respect to the yv plane; we shall always consider this half space), Q_2 has a minimum for $v^2 = Fr$, $x = 0$, $y = \ell$, $Q_2 = 2F\ell - w^2$, so that it can be negative only when $p > 1$.

Notice that for Q_1 or Q_2 equal to zero, a particular semi-permeable direction is $f = 0$, meaning that relative rest satisfies (5.1). The sign of Q_1 will turn out to be important in part of the analysis.

5.2 The Natural Barrier

i) The B.U.P. It is pointed out in [18] that the game can terminate only in the "usable part" of the capture set, such that, ν being the outward normal,

$$\min_{\varphi} \max_{\psi} \langle \nu, f(z,\varphi,\psi) \rangle < 0 \ .$$

For convenience, the capture set will now be considered as open, so that trajectories arriving tangentially to it still provide escape. Hence the strict inequality.

This usable part has a boundary given by

$$\min_{\varphi} \max_{\psi} \langle \nu, f(z,\varphi,\psi) \rangle = 0 \ .$$

Comparing this with equation (5.1), it is clear that we can attach to this line a semi-permeable surface having the same normal ν, and thus tangent to $\mathcal{C}$. Trajectories of this surface do not provide capture along the B.U.P.

This surface locally separates the state space into two regions. The first one contains the usable part of $\mathcal{C}$, and a game starting from a point of this region can be completed in a simple way. But from a point in the other region, if capture is possible the trajectory must in some sense go around the surface. Therefore, this surface is a barrier; it represents a discontinuity in the time of capture. This barrier emanating from the B.U.P. is called the natural barrier.

In our case, the boundary of the usable part is the curve $\mathcal{B}$ defined by

$$\min_{\varphi} \max_{\psi} [w(x \sin \varphi + y \cos \varphi) - vy] = w\ell - vy = 0$$

or

$$w - v \cos \theta = 0 \ .$$

It exists only for $v \geq w$. Projected on the yv plane, it appears as a hyperbola, extending from $v = w$, $y = \ell$ to infinity asymptotic to the v-axis.

ii) Equations of the Natural Barrier. At this point, we need to establish the differential equations resulting from (4.3) together with (5.1).

(5.1) gives, with the components of ν being ν_x, ν_y and ν_v

$$\min_{\varphi} \max_{\psi} \left[F\left(\frac{x\nu_y - y\nu_x}{v} \sin \varphi + \nu_v \cos \varphi\right) + w(\nu_x \sin \psi + \nu_y \cos \psi)\right]$$

$$- v\nu_y = 0 \ . \qquad (5.2)$$

Introduce the following notations:

$$\rho = \sqrt{\nu_x^2 + \nu_y^2}$$

$$\nu_\theta = y\nu_x - x\nu_y$$

$$\sigma = \sqrt{\frac{\nu_\theta^2}{v^2} + \nu_v^2} = \frac{1}{v}\sqrt{\nu_\theta^2 + v^2\nu_v^2} \ .$$

The strategies satisfying the minimax condition are given by (5.3), which shows that the evader points his velocity parallel to, and in the same direction as, (ν_x, ν_y)

$$\begin{aligned} \sin\varphi^* &= \frac{\nu_\theta}{v\sigma} & \qquad \sin\psi^* &= \frac{\nu_x}{\rho} \\ \cos\varphi^* &= \frac{\nu_v}{\sigma} & \qquad \cos\psi^* &= \frac{\nu_y}{\rho} \end{aligned} \tag{5.3}$$

and (5.2) becomes

$$H_1 = -F\sigma + \rho w - \nu_y v = 0 \ . \tag{5.2a}$$

The dynamical equations and the adjoint equations are then

$$\begin{aligned} \dot{x} &= -F\frac{y\nu_\theta}{v^2\sigma} + w\frac{\nu_x}{\rho} & \qquad \dot{\nu}_x &= -F\frac{\nu_y\nu_\theta}{v^2\sigma} \\ \dot{y} &= F\frac{x\nu_\theta}{v^2\sigma} + w\frac{\nu_y}{\rho} - v & \qquad \dot{\nu}_y &= F\frac{\nu_x\nu_\theta}{v^2\sigma} \\ \dot{v} &= -F\frac{\nu_v}{\sigma} & \qquad \dot{\nu}_v &= -F\frac{\nu_\theta^2}{v^3\sigma} + \nu_y \ . \end{aligned} \tag{5.4}$$

These equations will appear again; they actually are the Euler-Lagrange equations of the optimization problem. They are discussed in Appendix A. It is shown, in particular, that in the other coordinate system, their closed form integration, with any initial conditions, presents only elementary difficulties.

According to (5.2a), H_1 must be a first integral of (5.4); this can be checked directly from the equations. H_1 actually is the Hamiltonian of the "abnormal" optimization problem in the game of degree, as we shall see later.

We know the vector ν at the terminal point of the corresponding trajectories. Consequently, following the usual practice of dynamic programming, we integrate (5.4) backwards from the curve $\mathcal{B}$, calling τ the time to go. We choose the velocity as the parameter of $\mathcal{B}$, and to distinguish it from the running variable, call it s . Similarly, when needed, the corresponding angle θ will be labelled β . On $\mathcal{B}$ we have

$$x = \ell\sqrt{1-(w/s)^2} \qquad \nu_x = \rho\sqrt{1-(w/s)^2}$$

$$y = \ell(w/s) \qquad \nu_y = \rho(w/s)$$

$$v = s \qquad \nu_v = 0$$

where ρ is an arbitrary parameter, since the length of the vector ν is of no importance. Notice that an immediate consequence of equations (5.4) is the first integral

$$\rho = \text{constant.}$$

The solution is, for the half space $x \geq 0$,

$$\begin{aligned} x &= \frac{\sqrt{s^2-w^2}}{v}\left(\frac{1}{2}F\tau^2-w\tau+\ell\right) = \frac{\sqrt{s^2-w^2}}{v}Q(\tau) \\ y &= \frac{1}{v}\left[\frac{1}{2}F^2\tau^3 - \frac{1}{2}Fw\tau^2 + (s^2-w^2-F\ell)\tau + w\ell\right] \\ &= \frac{w-F\tau}{v}Q(\tau) + v\tau \\ v &= \sqrt{F^2\tau^2-2wF\tau+s^2} = \sqrt{(w-F\tau)^2+s^2-w^2} \end{aligned} \tag{5.5}$$

and for the adjoints

$$\begin{aligned} \nu_x &= \rho\frac{\sqrt{s^2-w^2}}{v} \\ \nu_y &= \rho\frac{w-F\tau}{v} \\ \nu_v &= -\rho\tau\frac{w-F\tau}{v} . \end{aligned} \tag{5.6}$$

The integrand in v has a minimum for $\tau = w/F$, and for this value our

formula gives

$$v^2 = s^2 - w^2 \geq 0$$

so that v is well-defined, as well as x and y .

Notice that x has the sign of $Q(\tau)$. As in the previous theory, the barrier closes in the (y,v) plane if and only if Q vanishes. And if $Q(\tau_1) = 0$, then our surface intersects the (y,v) plane along the straight line $\mathcal{L}$: $y = v\tau_1$. For $p = 1$, the surface is just tangent to the symmetry plane, and thus to the symmetric part, along that line, with $\tau_1 = w/F$.

It is interesting to compute the equations of a cross-section of this surface by a plane v = constant. We eliminate s between v and x , and obtain:

$$x = \frac{\sqrt{-F^2\tau^2+2wF\tau-w^2}}{v} Q(\tau)$$

$$y = \frac{w-F\tau}{v} Q(\tau) + v\tau .$$

We recognize equations (4.8), thus completely identifying this surface with the barrier already mentioned.

iii) <u>Termination</u>. However, although the trajectories (5.5) are smooth, sections at constant v of the surface (4.8) have a cusp for

$$\tau_c = \frac{1}{F}\left[w + \sqrt{\frac{1}{3}\,(w^2-2F\ell+2v)}\right]$$

value which is always larger than w/F , and thus than the lower root of $Q(\tau)$ if there is one, so that this cusp does not appear when the barrier closes.

The explanation of this cusp when the trajectories show no such anomaly is that the surface actually has a cusp, but the constituting trajectories are tangent to it, so that they are smooth across it. We have a verification of this fact by calculating the envelopes of the projections of the trajectories on two different planes, and checking that contact with the envelope is obtained for the same τ in the two

projections, so that the two envelopes are the projections of the same curve in the three-dimensional space.

It is lengthy but straightforward to see that formulas (5.5) give

$$\frac{\partial x}{\partial s}\frac{\partial v}{\partial \tau} - \frac{\partial v}{\partial s}\frac{\partial x}{\partial \tau} = s\,\frac{F\tau - w}{2v^2\sqrt{s^2-w^2}}\left[F^2\tau^2 - 2Fw\tau - 2(s^2-w^2-F\ell)\right]$$

$$\frac{\partial y}{\partial s}\frac{\partial v}{\partial \tau} - \frac{\partial y}{\partial \tau}\frac{\partial v}{\partial s} = \frac{s}{2v^2}\left[F^2\tau^2 - 2Fw\tau - 2(s^2-w^2-F\ell)\right],$$

which agrees with what we have just said.

This phenomenon is interesting in several respects. First, it shows that the barrier comes to an end. In fact, after the cusp, the surface is still semi-permeable, but with the vector ν pointing inside the capture region, so that it would correspond to a situation where the evader would be trying to force capture, against the will of the pursuer. This part must thus be discarded.

But also, it will appear that this is not an isolated case, but happens on most of our barriers. This case is the only one for which we have simple analytical formulas allowing a detailed analysis of the situation. A full understanding of the geometry of this case will help in other instances.

5.3 The Envelope Barrier

i) The Envelope Barrier. Another problem was pointed out by Isaacs. On $\mathcal{B}$ we have

$$\ell\ddot{r} = s^2 - w^2 - F\ell ,$$

so that for $s < \sqrt{w^2+F\ell}$ the trajectories (5.4) actually arrive at $\mathcal{B}$ from inside the capture circle, which they have thus penetrated at an earlier time. This is a typical occurrence of the problem pointed out in Chapter Two. As a consequence, these trajectories cannot be retained as escape trajectories.

We shall therefore consider $\mathcal{B}$ as interrupted at the point B :

$$v = s = \sqrt{w^2+F\ell} \qquad \cos\theta = \cos\beta = \frac{w}{\sqrt{w^2+F\ell}} \qquad r = \ell$$

and the crest at a point A' :

$$x = 0 \qquad y = \tau_o v \qquad v = \sqrt{w^2-F\ell}\ .$$

The barrier presents a "hole" at its lower v end, and what happens in that region is unanswered by the previous theory.

The way out of this difficulty was found by Isaacs: from B , one constructs a semi-permeable line of the lower dimensional game in which the state is constrained to remain on the capture circle. It is shown in [18] that this line has the following properties. Let $\mathfrak{D}$ be this line.

It is tangent to $\mathfrak{B}$ at B ;

It is such that a barrier can be constructed from it, made of trajectories that reach $\mathcal{C}$ tangentially to $\mathfrak{D}$;

This barrier, the "envelope barrier" $\mathfrak{E}$, provides a smooth extension to the natural barrier.

These facts can be understood in the following way. We know that at each point of the state space, there is a cone of possible semi-permeable directions. Taking a point on the non-usable part of the capture cylinder, we can find in this cone a direction (actually two) which is tangent to the cylinder. This defines a field of directions on the surface of the cylinder, equivalent to a differential equation. The curve $\mathfrak{D}$ is the integral of this equation through B .

Clearly, ν is normal to $\mathfrak{D}$ at each of its points. Therefore, the trajectories constructed with this ν form a barrier. By construction, they are tangent to $\mathfrak{D}$, and this is the only way in which a barrier can reach the non-usable part of $\mathcal{C}$ without penetrating it.

It is clear that once $\mathfrak{D}$ is reached, playing the strategies of the semi-permeable surface will cause the state to follow $\mathfrak{D}$ since they define a direction always tangent to it. Moreover, if we look a priori for trajectories straying on the surface of $\mathcal{C}$, $\mathfrak{D}$ appears as a semi-

permeable line, since $\mathcal{E}$ itself is semi-permeable.

Notice that $\mathfrak{D}$ is for the envelope barrier a cusp of the type discussed in the previous section, and the prolongation of the trajectories was, of course, discarded.

ii) <u>The I.R.G.</u> With our equations (4.5), the condition that a trajectory lie on $\mathcal{C}$ is

$$\dot{r} = w \cos (\psi-\theta) - v \cos \theta = 0$$

$$\cos (\psi-\theta) = \frac{v}{w} \cos \theta$$

$$\sin (\psi-\theta) = \frac{1}{w}\sqrt{w^2-v^2\cos^2\theta} \ ,$$

the sign of $\sin (\psi-\theta)$ being chosen in such a way that the evader runs away from the center line, toward the non-usable part.

The dynamics become:

$$\begin{aligned} \dot{\theta} &= -\frac{F}{v} \sin \varphi + \frac{1}{\ell} (v \sin \theta + \sqrt{w^2-v^2\cos^2\theta}) \\ \dot{v} &= F \cos \varphi \ . \end{aligned} \tag{5.7}$$

Referring to our vectorgram (Fig. 4), the requirement $\dot{r} = 0$ obliges the evader to choose his control at the point of Q which lies in the plane of P . Then, considering the restricted vectorgram in this plane, our geometrical theory shows that the pursuer must choose his control at the point of contact of one of the two possible tangent vectors. We choose the one that gives an increasing v to be in agreement with the natural barrier at B . It is found by maximizing the ratio $\dot{v}/\dot{\theta}$. Notice that it is clear from the geometry of the vectorgram (Fig. 4) that such tangents exist only if $Q_1 < 0$.

We introduce, following Isaacs,

$$a = \frac{F}{v} \qquad\qquad c = \frac{1}{\ell} (v \sin \theta + \sqrt{w^2-v^2\cos^2\theta}) \ .$$

The semi-permeable direction is obtained for

$$\sin \varphi = \frac{a}{c} \qquad\qquad \cos \varphi = \pm \frac{1}{c}\sqrt{c^2-a^2}$$

and the corresponding equations of motion are

$$\dot{\theta} = \frac{c^2-a^2}{c}$$

$$\dot{v} = F\frac{\sqrt{c^2-a^2}}{c} .$$

We can eliminate the time, avoiding a technical difficulty where $\dot{\theta} = \dot{v} = 0$:

$$\frac{d\theta}{dv} = +\frac{\sqrt{c^2-a^2}}{F} = \frac{\sqrt{\left(v \sin\theta + \sqrt{w^2-v^2\cos^2\theta}\right)^2 - \frac{F^2\ell^2}{v^2}}}{F\ell} . \qquad (5.8)$$

This equation does not seem to be integrable in closed form. Since in [18] only analytical solutions are sought, deliberately excluding numerical integrations, the problem is left at this point with the conjecture that $\mathfrak{D}$ might reach the (y,v) plane, whenever $p \geq 1$ at least, and the envelope barrier together with the natural barrier seal off a capture region. Escape would occur for any starting point outside this region.

It turns out that some more analytical results can be obtained. Then, the use of high-speed computers allows us to check them, and to proceed further with the investigation of the problem through numerical integration of the equations.

5.4 Termination of the Envelope Barrier

i) Termination of $\mathfrak{D}$. The curve $\mathfrak{D}$ is obtained by integration of (5.8) from B toward lower v's . This integration can be carried out, at least in principle, as long as $c^2 - a^2 \geq 0$. The question is whether it reaches the symmetry plane before reaching the surface $c^2 = a^2$. It cannot reach the plane $v = 0$ where c is finite and a infinite.

In the region of interest, both c and a are positive. Thus, we want

$$c \geq a \qquad \text{or} \qquad v \sin\theta + \sqrt{w^2-v^2\cos^2\theta} \geq \frac{F\ell}{v} .$$

In the vicinity of $\theta = 0$, $\frac{F\ell}{v} - v\sin\theta$ is positive. We isolate the square root and square both sides. This gives

$$Q_1 = \frac{F^2\ell^2}{v^2} + v^2 - 2F\ell \sin\theta - w^2 \leq 0 \ ,$$

which is consistent with our remark of the previous section, based upon the geometry of the vectorgram.

Let $\mathcal{Q}$ be the curve $Q_1 = 0$, $r = \ell$. It has a minimum in x for

$$v = \sqrt{F\ell} \qquad \sin\theta = 1 - \frac{w^2}{2F\ell} = 1 - p \ .$$

From this, we immediately conclude that

1) For $p < 1$, the curve $\mathfrak{D}$ never meets the symmetry plane $\theta = 0$;

2) For $p = 1$, if $\mathfrak{D}$ reaches $\theta = 0$, it is at the point A : $v = \sqrt{F\ell}$.

Thus, let us see in more detail what happens at that point.

Equation (5.8) shows that upon reaching $\mathcal{Q}$, $\mathfrak{D}$ has to be parallel to the v-axis. Thus, an integral can pass through A only if it has a curvature greater than that of $\mathfrak{D}$ at the same point.

Using its equation $Q_1 = 0$, we find for $\mathcal{Q}$,

$$\left(\frac{d^2\theta}{dv^2}\right)_{\mathcal{Q}} = \frac{\sin\theta}{F^2\ell^2\cos^3\theta}\left(v - \frac{F^2\ell^2}{v^3}\right)^2 + \frac{1}{F\ell\cos\theta}\left(1 + 3\,\frac{F^2\ell^2}{v^4}\right) .$$

At A , $\sin\theta = 0$, $v = \sqrt{F\ell}$, we obtain

$$\left(\frac{d^2\theta}{dv^2}\right)_{\mathcal{Q}} = \frac{4}{F\ell} \ .$$

Differentiating (5.8) with respect to θ , using $c = a$, one can check, after some rearrangements, that

$$\left(\frac{d^2\theta}{dv^2}\right)_{\mathfrak{D}} = \frac{1}{F\sqrt{w^2 - v^2\cos^2\theta}}\left[\frac{v\cos\theta}{F} + \sqrt{c^2-a^2}\left(\sin\theta + \frac{c\ell}{v}\right) - 2c^2\frac{\sin\theta}{\sqrt{c^2-a^2}}\right] .$$

When v goes to $\sqrt{F\ell}$, and θ to zero simultaneously with c^2-a^2 ,

this quantity satisfies, since $\sin\theta$ is positive,

$$\left(\frac{d^2\theta}{dv^2}\right)_{\mathfrak{D}} \le \frac{v\cos\theta}{F^2\sqrt{w^2-v^2\cos^2\theta}} = \frac{1}{F\ell} \ .$$

Therefore, no curve satisfying (5.8) can exist at A, and we have the following result:

Proposition: For $p = 1$, the curve $\mathfrak{D}$ does not reach the symmetry plane.

It has been found by numerical integration that it terminates at a point D, which is an equilibrium point of the relative motion, given by

$$v = 0.6002 \times w \qquad \theta = 0.0543 \text{ rd} \ .$$

Consequently, the envelope barrier does not seal the "hole" left by the natural barrier.

For values of p sufficiently larger, $\mathfrak{D}$ does close. The limiting value p_o has been numerically found to be

$$p_o \simeq 1.062 \ .$$

ii) The Envelope Barrier. To compute an incoming trajectory, we need the adjoint vector ν at each point of $\mathfrak{D}$. ν_v and ν_θ can be obtained from the fact that ν is normal to $\mathfrak{D}$. Then, the third component ν_r can be obtained from $H_1 = 0$ for the three-dimensional game.

In the cylindrical system of coordinates, we have

$$H_1 = -F\sqrt{\frac{\nu_\theta^2}{v^2} + \nu_v^2} + w\sqrt{\frac{\nu_\theta^2}{r^2} + \nu_r^2} + v\left(\frac{\nu_\theta}{r}\sin\theta - \nu_r\cos\theta\right) = 0 \ .$$

The first of the following two relations comes from the fact that ν is normal to $\mathfrak{D}$. Placing it in the above equation and rearranging, H_1 becomes a perfect square and yields the second relation:

$$\begin{aligned} \nu_v &= -\frac{\sqrt{c^2-a^2}}{F}\,\nu_\theta \\ \nu_r &= \frac{1}{\ell}\,\frac{v\cos\theta}{\sqrt{w^2-v^2\cos^2\theta}}\,\nu_\theta \ . \end{aligned} \tag{5.9}$$

It is convenient to introduce the parameter ρ

$$\rho = \sqrt{\nu_r^2 + \frac{\nu_\theta^2}{r^2}}$$

and the angle γ :

$$\nu_x = \rho \sin \gamma \qquad \nu_r = \rho \cos (\gamma-\theta)$$

$$\nu_y = \rho \cos \gamma \qquad \nu_\theta = r\rho \sin (\gamma-\theta)$$

and relation (5.9) yields γ , and consequently all three adjoints, through

$$w \cos (\theta-\gamma) = v \cos \theta \ . \tag{5.9a}$$

Notice that a consequence of this last relation is that at B , where $v \cos \theta = w$, $\theta = \gamma$. Therefore, $\nu_\theta = \nu_v = 0$ and $\nu_r = \rho$. We have a verification of the fact that ν is the same for both barriers at this point.

It has been found that the envelope barrier is terminated by a cusp of the type already described, that reaches the capture circle at D .

It is an interesting problem to find a way to characterize such a cusp with absolute certitude when the available data is numerical, and therefore approximate. In particular, it seems difficult to distinguish a cusp from a "fold" with very small radius of curvature.

The solution lies in the fact that together with the trajectories defining our surface, we compute, with a separate set of equations, the normal vector ν . This allows us to follow continuously a given side of the surface, and consequently to distinguish between a cusp and a finite radius of curvature, no matter how small (see Fig. 5).

A numerical localization of the cusp is possible with good accuracy by computing a curve on the surface, other than a trajectory.

The situation is now the following:

For $p < 1$: We have a smooth open barrier terminated by a cusp. Capture will occur from any initial condition, in agreement with

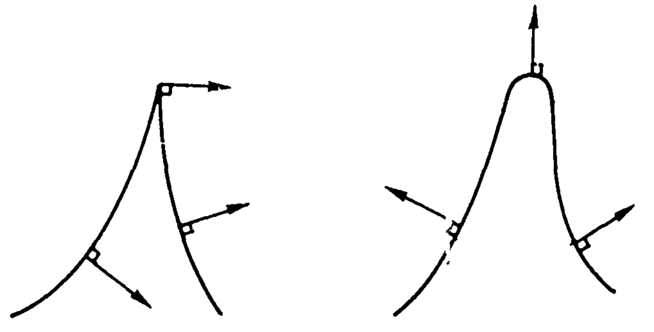

FIGURE 5. Characterization of a Cusp

the results of Chapter Four. The precise shape of the optimal capture trajectories will be investigated in Chapter Six.

For $p = 1$: The natural barrier closes forming a crest on the (y,v) plane. But the trajectories of the envelope barrier never reach that plane. Thus, the two symmetric parts of this barrier are tangent at A' and separate toward lower v's .

For $1 < p < p_0$: The envelope barrier forms a crest on part or all of its length. It is still open at the lower v end.

For $p \geq p_0$: The two barriers form a continuous surface that seals off a region of the state space. An evader coming from outside this capture region can always escape.

Figure 8 schematically depicts the situation for $p = 1$. The envelope barrier has been arbitrarily interrupted at y = constant for clarity.

5.5 The Envelope Junction

i) Motivation. The two barriers we know so far correspond to a chase in which the evader side-steps in an attempt to outmaneuver the pursuer. Curve $\mathfrak{D}$ terminates for small v's because the pursuer is then too maneuverable. We expect that the evader will take advantage of his greater speed and essentially flee from the pursuer.

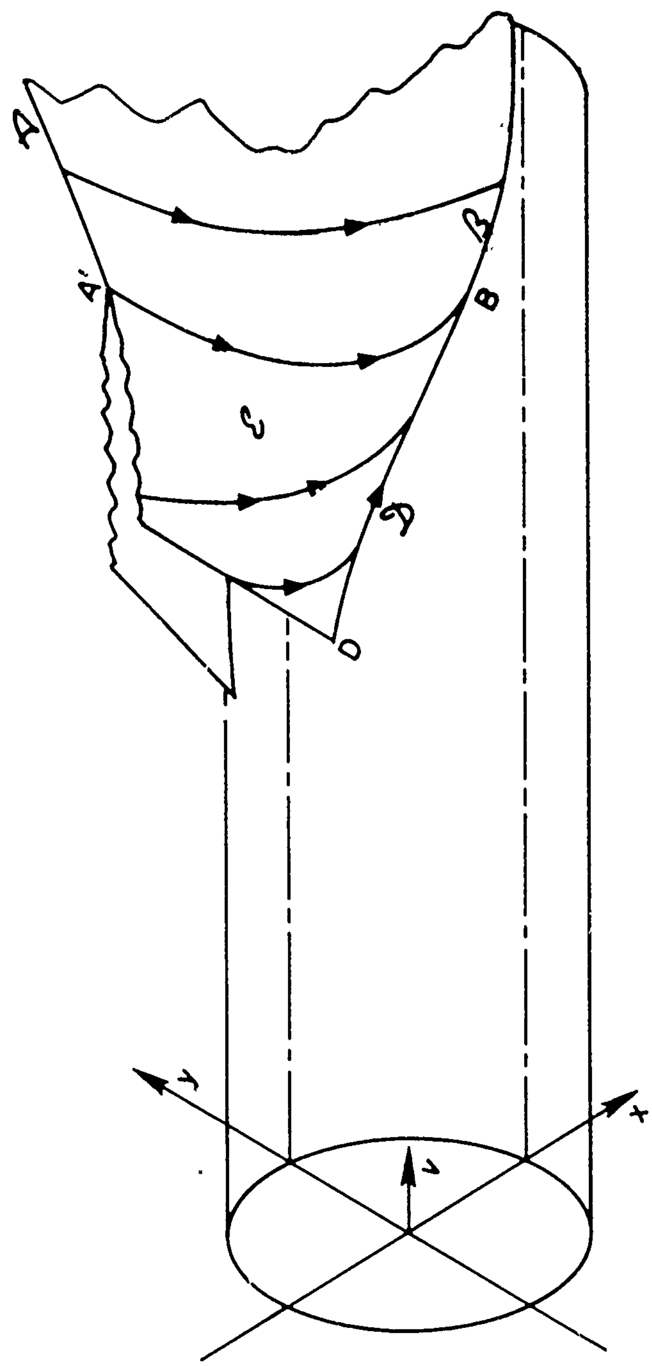

FIGURE 6. Envelope and Natural Barriers p = 1

Consider, in particular, a chase starting in the symmetry plane, at low v. It is intuitively clear that both players will direct their controls in the direction of the vector v. The ensuing motion, a straight chase in the physical space, will appear in the state space as a parabola in the (y,v) plane, the equations of which are given by

$$\frac{dy}{dt} = w - v \qquad\qquad \frac{dv}{dt} = F$$

that integrates into

$$y - y_o = \frac{1}{2F}\left[(w-v_o)^2 - (w-v)^2\right] .$$

If $p \geq 1$, the barriers reach the (y,v) plane. One of these parabolas just reaches the crest and provides escape. It should be part of a barrier, since a parabola immediately under it fails to reach the barrier.

For the worst case: $p = 1$, $y_o = \ell$ still corresponds to a positive v_o. Precisely, it must go through A', which gives

$$v_o = \frac{\sqrt{2}-1}{2} w .$$

We notice that such an escape trajectory, if it is to be retained as part of the barrier, presents a corner where the parabola reaches the crest. Hence the need for the equivalent of a corner condition for barriers. This is provided by the following theory.

ii) <u>A Corner Condition.</u> Let S^a and S^b be two semi-permeable surfaces intersecting at a non-zero angle. Each of them locally separates the space into two regions: R_1^a and R_2^a for S^a, and similarly for S^b, the subscripts being determined by the direction of ν (we purposely avoid specifying whether ν points into region one or two). The composite surface locally separates the space in two regions R_1 and R_2. Let us say that

$$R_1 = R_1^a \cap R_1^b \qquad\qquad \text{(dihedron less than } \pi \text{)}$$

$$R_2 = R_2^a \cup R_2^b \qquad\qquad \text{(dihedron more than } \pi \text{)} .$$

The composite surface S is obtained by discarding the portions of S^a and S^b lying in R_2. Let φ_1 be the control of the player who tries to go into region 1, and φ_2 the other player's control. We make the following assumption:

Assumption: On semi-permeable trajectories arriving at $J = S^a \cap S^b$, condition (5.1) uniquely defines φ_1.

Under this hypothesis we prove the following theorem:

Theorem: For S to be a barrier, it is necessary that the trajectories incoming to the junction do not cross it. They must either be tangent to it or present a corner.

Proof: Let us assume the contrary: some paths, say in S^a, actually cross J. Two situations can occur at J:

1) $\varphi_1^a \neq \varphi_1^b$. When the state reaches J, player 2 will keep his strategy φ_2^a. If player 1 keeps his strategy φ_1^a, by the current hypothesis he will let the state penetrate $R_2^b \subset R_2$. If he plays any other strategy, by our previous assumption he will let the state penetrate $R_2^a \subset R_2$. In every case the state penetrates R_2, and S is not a barrier.

2) $\varphi_1^a = \varphi_1^b$. Then when reaching J on S^a, player 1 has a control that prevents crossing of S^b, and consequently of J. Therefore, the trajectories of S^a cannot cross J.

This ends the proof.

Remark 1: $\varphi_1^a = \varphi_1^b$ is not necessary. A pair of strategies $(\varphi_1^a, \varphi_2^a)$ can generate paths reaching S^b tangentially without being equal to $(\varphi_1^b, \varphi_2^b)$ at J. In that case, the theorem states that the trajectories of S^a will fall back into R_1. Player 2 is thus obliged to change his strategy. He has the choice between two possibilities:

1) Either switch to φ_2^b; then player 1 will switch to φ_1^b, and the game will follow a trajectory of S^b, which supposedly leaves J;

2) Or vary φ_2 so as to be always in accordance with φ_2^a on the incoming trajectory at J. Player 1 must then choose the corresponding control φ_1^a, and the state will traverse J.

__Remark 2:__ J is for S^a a cusp of the type discussed earlier, so that S^a actually comes to an end on J. The same reasoning as in our proof applies to the junction of a barrier with the non-usable part of $\mathcal{C}$ and gives the envelope barrier. In that case, the surface we join on is not semi-permeable, and the evader has an infinity of strategies that prevent crossing it. But we needed to assume the unicity of φ_1 on the incoming trajectories only.

iii) __The "Roof" for $p = 1$__. For $p = 1$, the parabola we have described above is tangent to the barrier at A'. It is thus natural to construct an envelope junction from A' on the envelope barrier $\mathcal{E}$. As we do not have the analytical expression of the envelope barrier, finding the junction, say $\mathcal{J}$, as a barrier of the game constrained to lie on $\mathcal{E}$ is not feasible. We choose a different approach:

Let $f(\nu,z)$ be the direction defined by the controls verifying

$$\min_{\varphi} \max_{\psi} \langle \nu, f(z,\varphi,\psi) \rangle = \langle \nu, f(\nu,z) \rangle .$$

Let ν^b be the vector ν on $\mathcal{E}$. The problem is to find whether the equations

$$\begin{aligned} \langle \nu, f(\nu,z) \rangle &= 0 \\ \langle \nu^b, f(\nu,z) \rangle &= 0 \end{aligned} \tag{5.10}$$

have, for a given z, a solution $\nu^a \neq \nu^b$. The first equation says that ν belongs to the local cone of semi-permeable normals, and the second one that the corresponding direction is tangent to $\mathcal{E}$. We are looking, in the cone of semi-permeable directions, for a direction tangent to $\mathcal{E}$, other than that of the trajectory of $\mathcal{E}$.

If such a direction exists, for z in some neighborhood of A', we can consider, on the surface $\mathcal{E}$, the field of directions $f(\nu^a,z)$, and integrate it as a differential equation. As we are looking for a curve lying on the envelope barrier, which is known only numerically, carrying out this program presents some technical difficulties. The ideas of the numerical method used are outlined in Appendix B.

In our case, equations (5.10) can be made simple, introducing the

parameters ρ and γ as in the previous sections, and $\alpha = \theta - \gamma$, the angle between (ν_x, ν_y) and r . (Notice that $\psi^* = \gamma$.)

They become:

$$H_1 = -F\sigma + \rho(w - v \cos \gamma) = 0$$

$$-F(v^2 \nu_v^b \nu_v + \rho^2 r^2 \sin \alpha^b \sin \alpha) + \rho v^2 \sigma[w \cos (\gamma - \gamma^b) \qquad (5.11)$$

$$- v \cos \gamma^b] = 0 .$$

To avoid difficulties in the case $\nu_v^b = 0$, we solve the first equation for ν_v and put it in the second one. And we look for the roots in γ of the equation:

$$F^2 v \nu_v^b \sqrt{\frac{v^2}{F^2} (w - v \cos \gamma)^2 - r^2 \sin^2 \alpha} - F^2 \rho r^2 \sin \alpha^b \sin \alpha$$

$$+ v^2 \rho^2 (w - v \cos \gamma)[w \cos (\gamma - \gamma^b) - v \cos \gamma^b] = 0 \qquad (5.11a)$$

where, we recall,

$$\alpha = \theta - \gamma \qquad \alpha^b = \theta - \gamma^b \qquad (\rho^b \equiv \rho) .$$

It can be checked that $H_1^b = 0$ implies that this equation admits the root $\gamma = \gamma^b$, but we want a different one. At A' , $\gamma = 0$ is the desired root. For other points, this equation was solved numerically by Newton's technique.

We were able actually to compute a junction $\mathcal{J}$ and the corresponding semi-permeable surface $\mathfrak{R}$. The main feature is that $\mathcal{J}$ meets the cusp terminating $\mathfrak{E}$, and not $\mathfrak{D}$, at a point J :

$$x = 0.772 \ 10^{-2} \ \ell \qquad y = 1.37287 \ \ell \qquad v = 0.6438 \ w .$$

Consequently, the "roof" $\mathfrak{R}$, while it does seal the hole between $\mathfrak{E}$ and the symmetry plane, still leaves a "hole" in the barrier between the end of $\mathfrak{E}$ and the trajectory of $\mathfrak{R}$ arriving at J . This situation is depicted in Fig. 7.

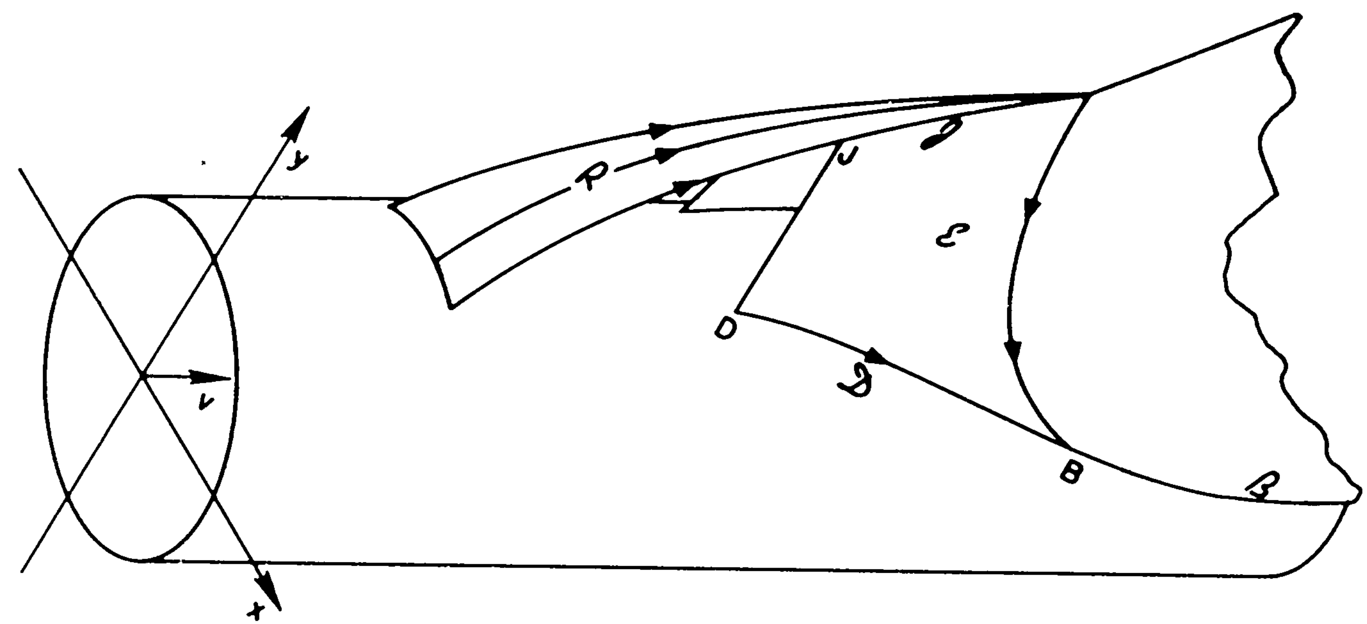

FIGURE 7. The Roof $\Re$ ($x \geq 0$ only is drawn) $p = 1$

5.6 Singular S.P.S.

i) Motivation. For $1 < p < p_2$, where $p_2 \simeq 1.092$, the envelope barrier forms a crest on part of its length, but the end of this crest has a smaller slope than the parabola at that point. Consequently, this parabola cannot be part of a barrier, since after reaching $\mathfrak{E}$ it falls back in the escape region, violating our necessary condition. Also, and this is closely related to the previous fact, a parabola immediately "under" it still reaches the barrier and provides escape, showing that the previous one was not a limiting escape trajectory.

Loosely speaking, on that parabola the evader was too strong. The limiting one, the one that verifies our theorem, is the parabola that reaches the crest tangentially, and this determines the point where the roof must attach. However, a new problem arises.

If from that point we apply the previous construction, we still find a junction $\mathfrak{J}$ and a corresponding roof $\mathfrak{R}$. But now, we start from a point where $\mathfrak{E}$ is not tangent to the symmetry plane. Consequently, the trajectory of $\mathfrak{R}$ reaching the crest does not lie in that plane, but reaches it at a non-zero angle. Therefore, the envelope roof is now made of two symmetric strips, leaving a hole between these two, in addition to the hole already described in the case $p = 1$.

In particular, we have not found what semi-permeable surface our parabola is imbedded in. To solve this problem, we need a slightly new concept.

ii) The Singular S.P.S. We have seen that, at each point of the state space there is a cone of semi-permeable directions. We claim that the family of trajectories generated by such a cone, by backward integration, is in fact a semi-permeable surface. We call such a surface a singular S.P.S., referring to the point where all its trajectories meet as its singular point.

That it is a semi-permeable surface can be seen by the fact that each vector ν at the singular point is normal to a tangent plane of the cone at this point. This is a consequence of our geometric construction

of this cone as the envelope of a family of planes Π normal to the ν's . Then, the transformation

$$\delta \dot{z} = \frac{\partial f}{\partial z} \delta z$$

transforms such a plane into a tangent plane at an ordinary point of the surface, and ν being governed by the adjoint equation will still have a constant, thus zero, dot product with δz .

This proof is based upon our geometrical theory of the semi-permeable cone. But the same fact can be seen in Isaacs' analytical theory, considering his theorem on the construction of semi-permeable surfaces ([18], theorem 8.3.1, p. 208). He parametrizes the initial curve with s and proves that

$$R_k = \sum_j \nu_j \frac{\partial z_j}{\partial s_k}$$

is a constant. If it was zero at $\tau = 0$, it remains zero for every τ . This can happen in two different ways:

either ν is normal to the line $z(s)$ (ordinary semi-permeable surface)

or $\partial z_i / \partial s = 0 \quad \forall\, i$ (singular semi-permeable surface) .

We are going to employ this concept to complete the roof.

iii) <u>Completion of the Roof</u>. Observe that at the end of the crest, incoming trajectories need not be tangent to $\mathcal{E}$. As long as their prolongation falls back into the capture region, as is the case for the parabola we retained, they do not violate our necessary condition.

From the point where we attached $\mathfrak{J}$, we can construct a singular surface, limiting it to those trajectories that are tangent to $\mathcal{E}$, and thus belong to $\mathfrak{R}$. All the trajectories of this family fall within the conditions of our theorem. This surface provides a smooth extension to the previously constructed roof, and completes it toward the symmetry plane.

Construction of the roof was carried out numerically for various values of the parameter. The shape of the surface obtained depends on this parameter. We found that two limiting values p_1 and p_2 have to be considered. Precise numerical determination of these values is difficult, due to low sensitivities. According to our calculations, they are in the ranges

$$1.056 < p_1 < 1.057 < p_o \simeq 1.062 < 1.091 < p_2 < 1.092 .$$

For $1 \leq p < p_1$, we have the same qualitative situation as for $p = 1$. The envelope junction reaches the cusp on $\mathcal{E}$. The roof seals the hole between the envelope barrier and the symmetry plane, but leaves two symmetrical ones between its last trajectory and the cusp on $\mathcal{E}$.

For $p_1 \leq p < p_2$, the envelope junction reaches the curve $\mathfrak{D}$. We have a closed capture region delineated by the natural barrier, the envelope barrier and the composite roof. Notice that the smallest value of the parameter for which the capture region is closed has been taken down from p_o to p_1 , closure, between these two values, being provided by the roof.

For $p_2 \leq p$, the crest has a slope larger than that of the parabolas on all of its length. No roof occurs. The natural barrier and the envelope barrier together define a closed capture region.

5.7 The Main Singular Barrier

i) Junction of Three Surfaces. From the point J where the envelope junction $\mathcal{J}$ meets the cusp on $\mathcal{E}$, we can generate another singular surface, $\mathcal{S}$, that provides a smooth extension to the roof toward the "side." The question of whether this semi-permeable surface, together with the rest of the barrier, still forms a barrier turns out to be difficult in two respects. The first problem has to do with what happens at J . It will be discussed here. The second one has to do with the intersection of $\mathcal{S}$ and $\mathcal{E}$. It will be mentioned at the end of this subsection, discussed in Section 5.8, and again at the end of Chapter Six.

At J , the trajectories of $\mathcal{S}$ seem to violate the necessary condi-

tion. Our theorem, however, was for the junction of two ordinary surfaces, and did not exclude the possibility of a third, singular one.

In this instance, it turns out that the direction f^a of the roof and the vector ν^s of $\mathcal{S}$ verify

$$\langle \nu^s, f^a \rangle > 0 .$$

It means that for this pair of surfaces, it is the pursuer who has the advantage of the larger region (region 2 of the theorem).

Then, upon arrival at J on a trajectory of $\mathcal{S}$, the sequence of decisions is as follows:

1) Since the trajectory extends into the escape region, the pursuer switches to the roof strategy.

2) If the evader does not switch, he will let the state drift below the roof. If he switches to the roof strategy, he places the state on a trajectory that goes back into the capture region. Thus he must choose the envelope barrier or the envelope junction strategy.

3) Thus the pursuer is obliged to switch again to counter the evader.

If the pursuer had switched to the envelope barrier strategy to start with, without the evader switching first, he would have let the state go above the roof in the escape region. On the other hand, the evader could not directly switch to the envelope barrier strategy, because then the pursuer would not have switched and the state would have gone under the singular barrier in the capture region.

The effect of the various choices can be shown in a diagram. We have plotted horizontally the three choices of the pursuer, by the name of the surface he plays according to, and vertically the choices of the evader. In each box, we have written P or E according to whether P or E wins with that combination. N stands for neutral. The arrows indicate the sequence we have described.

Several remarks must be made about this description.

$\psi \backslash \varphi$	$\mathcal{S}$	$\mathfrak{R}$	$\mathfrak{E}$
$\mathcal{S}$	E	P	E
$\mathfrak{R}$	E	P	E
$\mathfrak{E}$	P	E	N

First, notice that we are obliged to assume that the players know each other's control. This is not a very serious problem. One can, for instance, say that an infinitesimal loss is acceptable to them, and that the motion of the state during an infinitesimal time gives them the necessary information.

Observe also that the pursuer's last move could have been replaced by his going back to the singular barrier strategy. But then the evader would switch again, and we could have an infinite cycling, all supposed to be instantaneous! This is because what we have is a matrix game with no saddle point. We deliberately exclude the consideration of mixed strategies, which would anyway be of little help in a qualitative game. To solve this problem, we assume that both players prefer the neutral outcome to the risk of letting the opponent take the better. Then, the natural sequence of decisions is the one we proposed.

The second, and much more difficult, problem arises at this point. The envelope barrier trajectories can be considered as falling back into the capture region defined by the singular barrier. In fact, the P and the N in the last row of our matrix are not firmly established. This will be discussed in the next section.

ii) <u>Shape of $\mathcal{S}$.</u> We must investigate the qualitative shape of

$\mathcal{S}$, and see whether it seals the hole.

The first fact is that, whereas the trajectories close to the roof look like the roof and come from the capture cylinder, when we go to different enough directions (larger γ's), we find trajectories that do not come from the capture set. Moreover, we find that this surface, too, is terminated by a cusp, where the backward computation of those trajectories must be stopped. This cusp does not touch $\mathcal{C}$.

The situation at this point is depicted by Fig. 8, where we purposely avoided specifying what happens at the intersection of $\mathcal{S}$ and $\mathcal{E}$.

5.8 Another Envelope Barrier, Discussion

i) The Barrier $\mathcal{E}'$. If $\mathcal{S}$ is actually part of the barrier, it is easy to find yet another smooth extension to it. Let B' be the point where a trajectory of $\mathcal{S}$ is tangent to $\mathcal{C}$. Note that according to what we said in Section 5.4, B' has to lie in the region $Q_1 \leq 0$ since at that point there is a semi-permeable direction tangent to $\mathcal{C}$.

This direction can be imbedded in a family, as we argued when we constructed the curve $\mathcal{D}$. We must just take the other sign for φ , giving a decreasing v . We can in this way compute in $\mathcal{C}$ a curve $\mathcal{D}'$ given by equation (5.8) but with the opposite sign:

$$\frac{d\theta}{dv} = -\frac{\sqrt{\left(v \sin\theta + \sqrt{w^2 - v^2\cos^2\theta}\right)^2 - (F^2\ell^2/v^2)}}{F\ell} \qquad (5.8a)$$

and joining along this curve, a new envelope barrier $\mathcal{E}'$ can be constructed. The formulas for the ν's are the same, except for a minus sign in ν_v .

In the vicinity of B' , $\mathcal{E}'$ blends smoothly into $\mathcal{S}$, since the ν's are continuous across B' . It has been found, however, that starting at the point where the trajectory through B' touches the cusp, $\mathcal{E}'$ intersects $\mathcal{S}$ at a non-zero angle, cutting off the cusp. If we delete both surfaces beyond this intersection, we have a simple dispersal line of the game of kind.

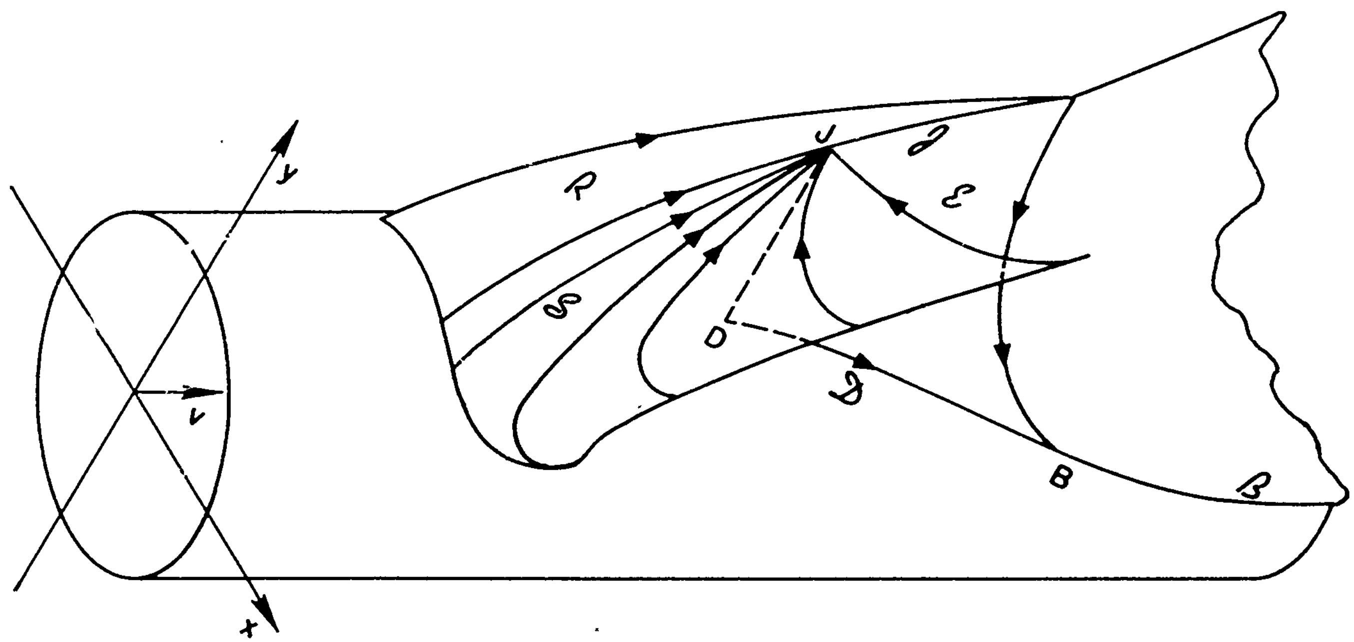

FIGURE 8. The Singular Barrier p = 1

$\mathfrak{E}'$ also intersects $\mathfrak{E}$, as $\mathfrak{D}'$ intersects $\mathfrak{D}$. Figure 9 is an attempt to describe this set of surfaces.

ii) Discussion . We have a system of semi-permeable surfaces apparently separating a closed capture region from an escape region. But the situation is more complicated than that.

As shown by Fig. 9, there is a common point to $\mathfrak{E}'$, $\mathcal{S}$ and the natural barrier. From that point to J , there must exist a line of intersection of $\mathcal{S}$ and the composite natural envelope barrier. The trajectories of $\mathfrak{E}$ penetrate $\mathcal{S}$ along this intersection, and the necessary condition of Section 5.5 is violated. Therefore, the set of surfaces we have described does not constitute a barrier.

The problem cannot be solved by discarding the part of $\mathfrak{E}$ "above" $\mathcal{S}$, as the existence of $\mathcal{J}$ and $\mathcal{S}$ itself is based upon the existence of $\mathfrak{E}$. A conjecture will be presented later as to how the barrier may look. Using the game of degree, a part of $\mathcal{S}$ would be discarded. $\mathfrak{E}'$ would be kept complete as a barrier, and not truncated where it intersects $\mathcal{S}$.

Then we still have trajectories of $\mathfrak{E}$ crossing the intersection with $\mathfrak{E}'$. However, this is not a contradiction for the following reason: along this intersection, the "escape region" is the region outside of both $\mathfrak{E}$ and $\mathfrak{E}'$. When seen as such, the intersection has only trajectories leaving it. The part of $\mathfrak{E}$ inside $\mathfrak{E}'$ merely defines the region for which $\mathfrak{E}'$ is a barrier. Inside of $\mathfrak{E}$, $\mathfrak{E}'$ does not exist.

This conjecture is depicted by Fig. 10.

5.9 Conclusion

Our investigation of the game of kind can be summarized as follows:

$p < 1$ The natural barrier and the envelope barrier together form an open barrier terminated by a cusp. Capture occurs from any initial condition.

$1 \leq p < p_1$ We have not been able to display a barrier sealing off a region of the state space. We conjecture that an open bar-

rier exists, made of six intersecting semi-permeable surfaces, plus their image in the symmetry plane. Following these surfaces could involve a six-stage chase.

$\underline{p_1 \leq p}$ We have a closed capture region; an evader starting far enough away will always escape. The configuration of the barrier depends on the relative values of p and p_2 :

$\underline{p < p_2}$: The barrier involves the natural barrier, the envelope barrier, the envelope roof and the singular roof.

$\underline{p_2 \leq p}$: The roof no longer exists. The closed barrier is made of the two surfaces found by Isaacs.

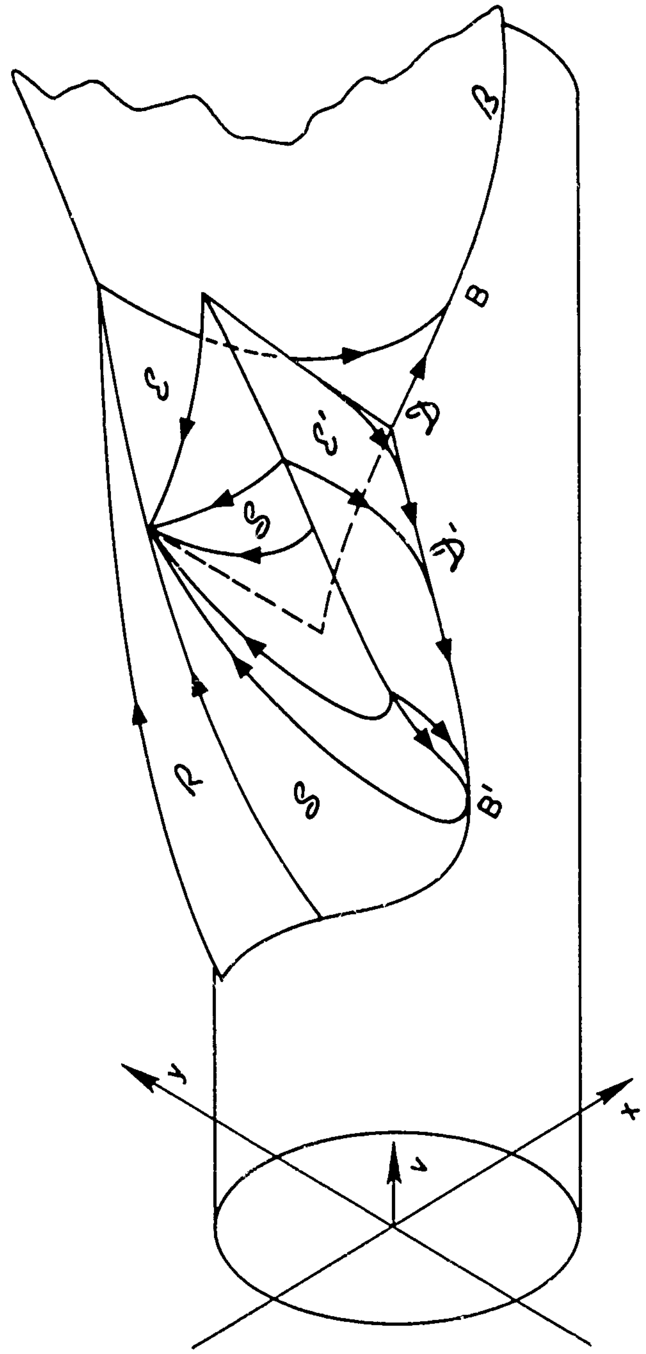

FIGURE 9. All the Barriers p = 1

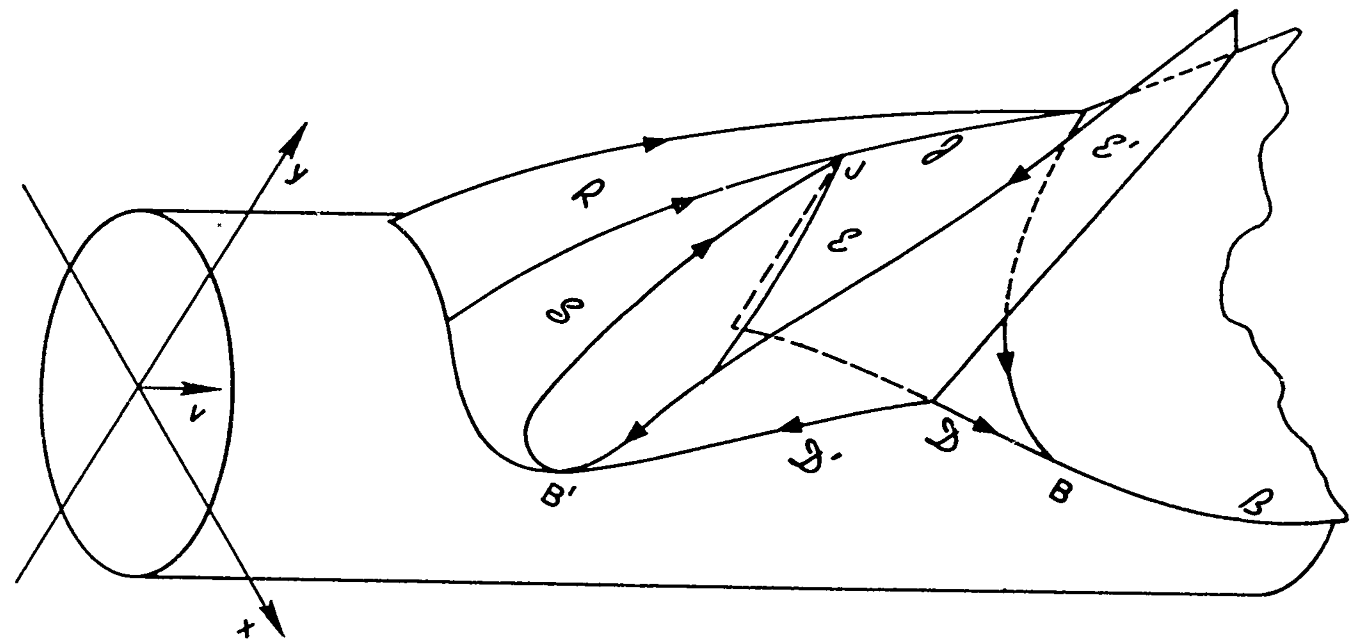

FIGURE 10. Conjectured Configuration of the Barrier $p = 1$

6. THE GAME OF DEGREE

The previous chapter was concerned with the qualitative problem: capture or escape. It yielded surfaces of discontinuity of the qualitative problem or "game of degree" we want to investigate now. What are the optimal strategies and the corresponding capture time and trajectories?

6.1 The Hamilton-Jacobi Equation

i) The Problem. Let the capture region as defined by the previous chapter be $\mathcal{C}'$. When the barrier is open, $\mathcal{C}'$ is the whole state space, deprived of the capture set.

We are looking for a pair of strategies $\varphi^*(z)$, $\psi^*(z)$ defined in the capture region, and such that

1) $\varphi^*(z) \in \Phi \quad \psi^*(z) \in \Psi \quad \forall z \in \mathcal{C}'$

2) The equation of motion

   $\dot{z} = f(z, \varphi^*(z), \psi^*(z))$

   has a solution, not necessarily unique, lying in $\mathcal{C}'$ for every $z(0) \in \mathcal{C}'$

3) These solutions transfer $z(0)$ to the terminal manifold $\mathcal{C}$ in a finite time, and yield a uniquely determined payoff $J(z, \varphi^*, \psi^*) = V(z)$, (in our case, J is simply the time of capture), verifying

   $$\min_{\varphi(\cdot)} \max_{\psi(\cdot)} J(z,\varphi,\psi) = \max_{\psi(\cdot)} \min_{(\cdot)} J(z,\varphi,\psi) = V(z)$$

The main tool used in the solution of this problem is a generalized version of the Hamilton-Jacobi equation. It was first derived by Isaacs who called it the "main equation". His derivation can be found in [18]. Berkovitz [1] gave a more rigorous derivation, using a variational technique. Several other authors gave proofs of varying generality. A recent one we already referred to can be found in Blaquière Gérard, and

Leitmann [3] uses Leitmann's geometrical theory of optimal process, and is similar to Isaacs' second proof.

We shall give a simple derivation, valid only under too strong assumptions. It follows Carathéodory's method, as did Chattopadhay [10] for a slightly different case. Our aim is to point out that the existence of a saddle point in the Hamiltonian is sufficient to insure that the game itself has a saddle point.

We directly arrive at a sufficient condition, reached by Isaac through his verification theorem, and by Berkovitz through a generalization of Hilbert's invariant. Notice that Berkovitz' technique being variational yields the Euler-Lagrange equations. Hence the need of Hilbert's invariant to show that they are the characteristics of a partial differential equation. In every case sufficiency arises from the consideration of a field of extremals. (See [5]).

ii) Derivation. We consider the more general case of a non-stationary, integral payoff game, defined by

$$\dot{z} = f(z,\varphi,\psi,t)$$

$$J = K(z,t)\Big|_{z\in\mathcal{C}} + \int_{t_o}^{t_f} L(z,\varphi,\psi,t)\, dt \qquad (6.1)$$

where t_f is the first instant such that $z(t)\in\mathcal{C}$, the terminal manifold (possibly time varying).

We define the Hamiltonian function

$$H(z,\lambda,\varphi,\psi,t) = L(z,\varphi,\psi,t) + \langle\lambda, f(z,\varphi,\psi,t)\rangle$$

Assume that

$$\min_{\varphi}\ \max_{\psi} H(z,\lambda,\varphi,\psi,t) = \max_{\psi}\ \min_{\varphi} H(z,\lambda,\varphi,\psi,t) = H^*(z,\lambda,t)$$

and that this extremum is attained for a uniquely defined pair of controls, except, possibly, on some singular manifolds. Let these controls be

$$\varphi^* = \varphi^*(z,\lambda,t) \qquad \psi^* = \psi^*(z,\lambda,t)$$

verifying conditions 1) and 2) above.

Consider the Hamilton-Jacobi equation

$$\frac{\partial V}{\partial t} + H^*\left(z, \frac{\partial V}{\partial z}, t\right) = 0 \tag{6.2}$$

Theorem: Under the above assumptions, if the Hamilton-Jacobi equation (6.2) has a continuously differentiable solution $V(z,t)$ the restriction of which to $\mathcal{C}$ verifies

$$V(z,t)\big|_{\mathcal{C}} = K(z,t)$$

and if the restrictions to a suitable interval (t_o, t_f) of the strategies

$$\varphi^*\left(z, \frac{\partial V}{\partial z}, t\right) \qquad \psi^*\left(z, \frac{\partial V}{\partial z}, t\right)$$

transfer (z_o, t_o) to $\mathcal{C}$ at t_f, then these strategies are optimal in the sense of section 6.1, and the corresponding payoff is $V(z_o, t_o)$.

Proof: To prove this result, we first establish Carathéodory's lemma:

Lemma: Let the scalar function $N(z,\varphi,\psi,t)$ have, for every z and t a unique saddle point equal to zero, at $\varphi^*(z,t)$ and $\psi^*(z,t)$

$$N(z,\varphi,\psi^*(z,t),t) > 0 \qquad \forall\ \varphi \in \Phi, \quad \varphi \neq \varphi^*(z,t)$$

$$N(z,\varphi^*(z,t),\psi^*(z,t),t) = 0$$

$$N(z,\varphi^*(z,t),\psi,t) < 0 \qquad \forall\ \psi \in \Psi, \quad \psi \neq \psi^*(z,t)$$

and assume that (φ^*,ψ^*) transfers the state (z_o, t_o) to $\mathcal{C}$ at t_f, then the game with payoff

$$I = \int_{t_o}^{t_1} N(z,\varphi,\psi,t)\,dt \qquad z(t_1) \in \mathcal{C}$$

admits (φ, ψ) as optimal strategies for the initial conditions (z_o, t_o), and the optimal payoff is $I = 0$.

Proof of the Lemma. It is a simple matter to check that, for any t_1,

$$I(z_o, \varphi^*, \psi, t_o) = \int_{t_o}^{t_1} N(z, \varphi^*, \psi, t)\,dt < 0$$

$$I(z_o, \varphi^*, \psi^*, t_o) = \int_{t_o}^{t_f} N(z, \varphi^*, \psi^*, t)\,dt = 0$$

$$I(z_o, \varphi, \psi^*, t_o) = \int_{t_o}^{t_1} N(z, \varphi, \psi^*, t)\,dt > 0$$

hence I verifies (6.2) and the lemma is proved. Then, introduce

$$N = L + \frac{\partial V}{\partial t} + \left\langle \frac{\partial V}{\partial z}, f \right\rangle$$

where $V(z,t)$ verifies (6.2). Clearly, by definition of H^*, N verifies the assumptions of the lemma.

Next, observe that along a trajectory, $N = L + \frac{dV}{dt}$, so that $\frac{dV}{dt}$ being continuous

$$I = \int_{t_o}^{t_f} N\,dt = V(z_f, t_f) - V(z_o, t_o) + \int_{t_o}^{t_f} L\,dt$$

and since $V(z_f, t_f) = K(z,t)$, we have

$$I = J - V(z_o, t_o)$$

As V is a function of time and state only, and not of φ and ψ, finding the minimax of I for a given (z_o, t_o) is equivalent to finding that of J. We apply Carathéodory's lemma to I, and recall that its optimal value is zero. We immediately obtain:

J has a saddle point
The optimal strategies are φ^* and ψ^*
The optimal payoff is $J = V(z_o, t_o)$

which proves the theorem.

Remark: In our case, f, and consequently H, is separated:

$$H = H_E(z,\psi) - H_P(z,\varphi)$$

so that we are assured that it has a saddle point.

It is well known that equation (6.2) can be solved using the method of characteristics, which yields the Euler-Lagrange, or canonical, equations

$$\begin{aligned} \dot{z} &= \frac{\partial H^*}{\partial \lambda} = \left.\frac{\partial H}{\partial \lambda}\right|_{\varphi=\varphi^*,\,\psi=\psi^*} \\ \dot{\lambda} &= -\frac{\partial H^*}{\partial z} = -\left.\frac{\partial H}{\partial z}\right|_{\varphi=\varphi^*,\,\psi=\psi^*} \end{aligned} \tag{6.3}$$

where, as we shall do from now on, we represent the gradient of V by the symbol λ. We call its components the adjoint variables.

It is interesting to notice that the semipermeable condition (5.1) can be regarded as the limit of (6.2), where $\partial V/\partial t \equiv 0$, when the gradient of V is infinite. Then, the term in $\langle V_z, f\rangle$ is predominant, and by rescaling to a finite ν such that $\lambda = \nu/\nu_o$, and letting ν_o go to zero, the term in L disappears.

This corresponds to nonregular points of the linear theory, and to the classical abnormal problem of calculus of variations.

6.2 The Primaries

i) A Compact Form. We use representation (4.2) (see[4]). The Hamiltonian is:

$$H = 1 - \vec{\lambda}_r \cdot \vec{v} + w\vec{\lambda}_r \cdot \hat{\alpha} + F\vec{\lambda}_v \cdot \hat{\beta}$$

the saddle point is obtained for

$$\hat{\alpha} = \frac{\vec{\lambda}_r}{\rho} \quad \hat{\beta} = -\frac{\vec{\lambda}_v}{\sigma} \tag{6.4}$$

where $\rho = |\vec{\lambda}_r|$ $\sigma = |\vec{\lambda}_v|$. This yields

$$H^* = 1 - F\sigma + \rho w - \vec{\lambda}_r \cdot \vec{v}$$

The adjoint equations are

$$\begin{aligned} \dot{\vec{\lambda}}_r &= 0 \qquad & \vec{\lambda}_r &= \text{constant} \\ \dot{\vec{\lambda}}_v &= \vec{\lambda}_r \qquad & \vec{\lambda}_v &= \vec{\lambda}_{v_o} - \vec{\lambda}_r \tau \end{aligned} \tag{6.5}$$

where $\vec{\lambda}_{v_o}$ is the adjoint vector $\vec{\lambda}_v$ at time of capture, and τ the time to go $\tau = t_f - t$.

Now, the capture set $\mathcal{C}$: $\vec{r}^{\,2} = \ell^2$ is a surface of constant payoff $\tau = 0$. Therefore the gradient of τ is normal to $\mathcal{C}$, which gives

$$\vec{\lambda}_r = \frac{\rho}{\ell} \vec{r}_o \qquad \vec{\lambda}_{v_o} = 0 \tag{6.6}$$

where ρ is given by $H = 0$. At $\tau = 0$, let

$$\vec{\lambda}_r \cdot \vec{v}_o = \rho v_o \cos\beta$$

so that $H = 0$ gives

$$\rho = \frac{1}{v_o \cos\beta - w}$$

which is consistent with our claim that along the B.U.P. the adjoint vector is infinite.

The consequence of equations (6.5) and (6.6) is that $\vec{\lambda}_r$ and $\vec{\lambda}_v$ have a constant direction, $\vec{\lambda}_v$ being parallel, and opposite, to $\vec{\lambda}_r$. This together with (6.4) gives the first important result on the game of degree:

In an optimal chase, the evader runs in a straight line in the physical space, and the pursuer keeps his acceleration parallel to the evader's velocity, describing a parabola.

We integrate the equations of motion with the calculated optimal controls. (The subscript zero stands everywhere for the time $\tau = 0$, $t = t_f$)

$$\vec{v} = \vec{v}_o - F\hat{\beta}\tau$$

$$\vec{r} = \vec{r}_o + \vec{v}_o\tau - w\hat{\beta}\tau - F\hat{\beta}\frac{\tau^2}{2}$$

taking into account that $\vec{r}_o = \ell\hat{\beta}$, we obtain

$$\vec{r} = \vec{v}_o\tau + \left(\ell - w\tau - F\frac{\tau^2}{2}\right)\hat{\beta}$$

we can eliminate $\vec{v}_o$ in terms of $\vec{v}$, we find

$$\vec{r} - \vec{v}\tau = Q(\tau)\hat{\beta} \qquad Q(\tau) = \frac{1}{2}F\tau^2 - w\tau + \ell$$

we recognize equation (4.7a), thus identifying the time to go along those trajectories, the primaries, with the estimating function of the previous theory.

ii) <u>3-D Representation</u>. To investigate the shape of our trajectories, it is convenient to come back to the three dimensional representation.

As our Lagrangian L is constant, maximizing H or H_1 is the same problem, and the Euler-Lagrange equations (6.3) are the same as in the game of kind.

Thus our trajectories are still solutions of (5.4). The only differences are that v must be replaced by λ, and that ρ, still constant along a trajectory, is no longer arbitrary but defined by

$$\rho = \frac{1}{s \cos \beta - w} .$$

The detailed treatment of these equations is given in Appendix A. It is convenient, to express their solution, to introduce the parameters

$$\xi = s \sin \beta$$

$$\eta = s \cos \beta \qquad \rho = \frac{1}{\eta - w}$$

and the equations of the trajectories and of the adjoints are:

$$\begin{aligned} x &= \frac{\xi}{v} Q(\tau) & \lambda_x &= \rho \frac{\xi}{v} \\ y &= \frac{\eta - F}{v} Q(\tau) + v\tau & \lambda_y &= \rho \frac{\eta - F\tau}{v} \\ v &= (\xi^2 + (\eta - F\tau)^2)^{1/2} & \lambda_v &= -\rho\tau \frac{\eta - F\tau}{v} \end{aligned} \tag{6.7}$$

Notice that as expected

$$x^2 + (y - v\tau)^2 = Q(\tau)^2$$

Notice also that equations (5.5) appear as a special case of these with $\xi = \sqrt{s^2 - w^2}$ and $\eta = w$.

For $p \geq 1$, all the primaries meet on the line $\mathcal{a}$: $x = 0$, $y = v\tau_1$ where $Q(\tau_1) = 0$. In the region where the natural barrier exists, this line is its crest. From each point of it there is an infinity of optimal trajectories yielding the same capture time. This line is similar

to the point A of the Homicidal Chauffeur game. (See [18], [17]). But for $v < \sqrt{w^2 - F\ell}$, that is beyond the point A' toward the lower v's, this line is under the roof, inside the capture region. There, it represents a line of focal points. It is known that beyond such a point the trajectories are no longer optimal (see [7]). We shall, later on, propose a solution to this problem.

We must investigate whether the field of primaries presents any other singularity, in particular in the case $p < 1$ where $\mathcal{A}$ does not exist. We calculate the functional determinant:

$$\Delta = \frac{D(x,y,v)}{D(\tau,\xi,\eta)}$$

This determinant is

$$\Delta = \begin{vmatrix} \frac{\xi}{v}\left[F\tau - w + F\,\frac{\eta - F\tau}{v^2}\;Q(\tau)\right] & -\frac{1}{v}\left[\frac{\xi^2}{v^2}\,FQ(\tau) + w(\eta - F\tau)\right] + v & -f\,\frac{\eta - F\tau}{v} \\ \frac{(\eta - F\tau)^2}{v^3}\,Q(\tau) & \frac{\xi}{v}\left(\tau - \frac{\eta - F\tau}{v^2}\;Q(\tau)\right) & \frac{\xi}{v} \\ -\frac{\xi}{v^3}\,(\eta - F\tau)\,Q(\tau) & \frac{1}{v}\,(F\tau - w)(\eta - F\tau) + v & \frac{\eta - F\tau}{v} \end{vmatrix}$$

We first look at the case $\eta - F\tau = 0$, $v = \xi$, which gives

$$\Delta = \begin{vmatrix} \eta - w & \xi - F\,\frac{Q(\tau)}{\xi} & 0 \\ 0 & \tau & 1 \\ 0 & \frac{Q(\tau)}{\xi} & 0 \end{vmatrix} = -\,\frac{Q(\tau)}{v}\,(\eta - w)$$

For $\eta \neq F\tau$, we add some linear combinations (involving the coefficient $\xi/\eta - F\tau$) of some lines to others, yielding the same value:

$$\Delta = \begin{vmatrix} \frac{\xi}{v}(F\tau - w) & \frac{1}{v}(F\tau-w)(\eta-F\tau)v & 0 \\ \eta - F\tau & -\xi & 0 \\ 0 & \tau & 1 \end{vmatrix} = -\frac{Q(\tau)}{v}(\eta - w)$$

The primaries correspond to $\eta > w$, thus this determinant is zero only for $Q(\tau) = 0$. For $p < 1$, it is never zero, the field has no singularity.

The second question to ask is whether this field fills the whole space. It was seen in Section (4.4) that for every (x,y,v) there corresponds a τ, and once τ is known, it is straightforward to deduce from (6.7) that

$$\xi = \frac{vx}{Q(\tau)}$$

$$\eta = \sqrt{v^2 - \xi^2} + F\tau$$

and τ being a root of

$$x^2 + (y - v\tau)^2 = Q(\tau)^2$$

we see that $x^2/Q(\tau)^2$ is greater than one, and consequently that $v^2 - \xi^2$ is always positive.

6.3 The State Constraint

i) The Concept. Having identified τ along the primaries with the estimating function, it is natural to check whether the corresponding trajectories penetrate the capture set, as suggested by the behavior of the barrier.

Calculating r^2 we find

$$r^2 - \ell^2 = \tau P(\tau) = \tau\left[\frac{1}{4}F^2\tau^3 - F(\eta-w)\tau^2 + \left((\eta-w)^2 + \xi^2 - F\ell\right)\tau + 2\ell(\eta-w)\right]$$

as expected, $r^2 = \ell^2$ for $\tau = 0$. But the polynomial $P(\tau)$ can have a positive root. Actually, since $P(0)$ is positive, it has two such roots or none. The limiting case corresponds to a trajectory tangent to $\mathcal{C}$.

A whole portion of the field must be discarded because of this fact. It would correspond to trajectories penetrating $\mathcal{C}$ and leaving it again before capture. This is sketched in Fig. 11, in which the shaded region is left unaccounted for by the present construction.

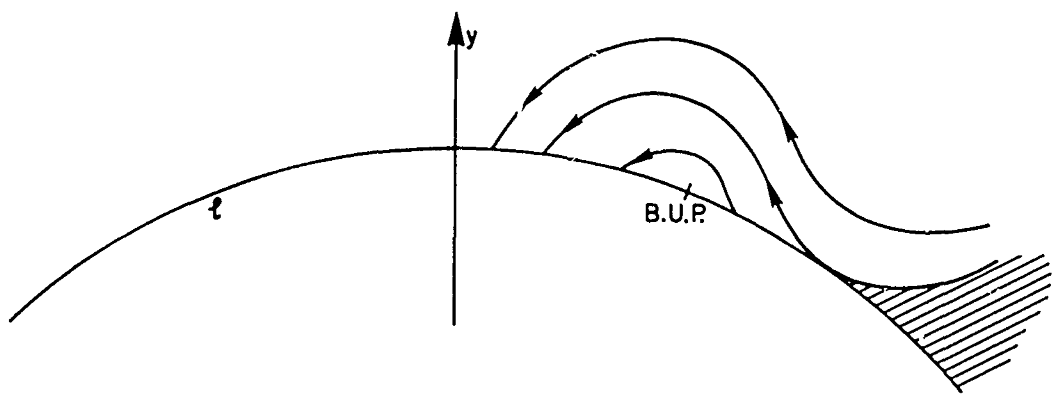

FIGURE 11. The Primaries

This problem was discovered by J. V. Breakwell and Boardman; see [4]. The solution is their concept of "safe contact," that appears in other games as well. (See [6] and [20]).

Let us consider the general game (6.1). We have to discard trajectories because they penetrate $\mathcal{C}$ at a nonzero τ. If this happens in the usable part of the capture circle it raises no problem since in that region we know another strategy that forces immediate capture anyway. Let us consider the case where this happens in the nonusable part of the capture set. Then, the region where we have no trajectory is adjacent to this nonusable part of $\mathcal{C}$, where E can prevent immediate capture.

In that part of $\mathcal{C}$, E can in particular maintain the state at the surface of the capture set, without penetrating it. One can in principle chose a coordinate system such that $\mathcal{C}$ be the set $z_n < \mathcal{C}$, and the "safe contact condition" is:

$$\dot{z}_n = f_n(z,\psi,\psi) = 0 \tag{6.8}$$

(We restrict ourselves to a first order constraint. See [8].) and this can be viewed as a relation that ψ has to satisfy. We assume that the optimal paths include a leg of this type, leaving $\mathcal{C}$ the first time they meet a trajectory of the n-dimensional game doing so. We check a posteriori that this allows us to construct a field of extremals filling the previously empty region, and to which the verification theorem applies.

Consider the reduced game in the (n-1) first state variables on the surface of $\mathcal{C}$. We know its sensitivity vector at the point where the trajectories already known leave $\mathcal{C}$: since $V(z)$ has to be uniquely defined, it is the projection of the n-dimensional sensitivity vector at the same point. (See the "jump condition" in [3] and [9]). Therefore, we can integrate constrained trajectories backward from these points, together with their adjoints.

Then, at each point of a constrained path, we can compute a incoming extremal of the unconstrained game, recovering the missing adjoint by the main equation $H = 0$. If this construction actually yields a field of trajectories filling the void region, we have reached our objective. According to Issacs' terminology, the extremals of the n-

dimensional game constructed in this way will be called the "tributaries" of the constrained trajectories.

Notice the obvious similarity between the construction and the one that led to the envelope barrier. The curve $\mathcal{D}$ now appears as a natural limit of the constrained game, and the envelope barrier itself as the corresponding family of tributaries. About this construction, we prove the following simple result.

Definition. The state constraint (6.8) is said to be singular if the equation

$$f_n(z,\psi^*(z,\lambda),\ \psi^*(z,\lambda) = 0 \tag{6.8a}$$

cannot be solved for λ_n in terms of the (n-1) first λ_k's, and z. For a further discussion of the terminology and of the relation with known results of calculus of variations, see Appendix D.

Theorem. If the safe contact condition is not singular, the state constraint is reached (and left) tangentially.

Proof. Let $\hat{z} = (z_1,\ldots, z_{n-1})$, $\hat{\lambda} = (\lambda_1 \ldots \lambda_{n-1})$, and $\hat{\psi}$ denote a control ψ satisfying the safe contact condition. The dynamics of the reduced game are

$$\dot{\hat{z}} = \hat{f}(\hat{z},\varphi,\hat{\psi})$$

Its optimal strategies are given by the functions $\varphi^o(\hat{z},\hat{\lambda})$ and $\hat{\psi}^o(\hat{z},\hat{\lambda})$ verifying

$$\min_{\varphi}\max_{\hat{\psi}}\left(L(\hat{z},\psi,\hat{\psi}) + \sum_1^{n-1}\lambda_k\hat{f}_k(\hat{z},\varphi,\hat{\psi})\right) = \hat{L}(\hat{z},\varphi^o,\hat{\psi}^o)$$

$$+ \sum_1^{n-1}\lambda_k\hat{f}_k(\hat{z},\varphi^o,\hat{\psi}^o) \tag{6.9}$$

and moreover, this quantity is $\hat{H}^o = 0$.

The n^{th} component of λ in the n-dimensional game, at any point of the constrained game is given by $H^* = 0$, where

$$H^* = \min_{\varphi} \max_{\psi} \left(L(z,\varphi,\psi) + \sum_1^n \lambda_k f_k(z,\varphi,\psi) \right) = L(z,\varphi^*,\psi^*)$$

$$+ \sum_1^n \lambda_k f_k(z,\varphi^*,\psi^*) \qquad (6.9a)$$

Under our hypothesis, we have a simple way to find a solution of this equation. Let λ_n satisfy

$$f_n(z,\varphi^*(z,\lambda),\ \psi^*(z,\lambda) = 0$$

For this λ, $f_k(z,\varphi^*,\psi^*) = \hat{f}_k(z,\varphi^*,\psi^*)$ for every $k < n$. Consequently, (6.9) and (6.9a) are identical, and yield identical implicit functions. Once z_n and λ_n are known:

$$\varphi^o(\hat{z},\hat{\lambda}) = \varphi^*(z,\lambda)$$

$$\hat{\psi}^o(\hat{z},\hat{\lambda}) = \psi^*(z,\lambda)$$

and therefore

$$H^* = \hat{H}^o + \lambda_n f_n(z,\varphi^*,\psi^*) = 0$$

which proves that this λ is the solution sought. Then, the equality of the optimal controls on both arcs proves the theorem.

The proof can easily be generalized to a multivariable state constraint. It does not specify whether the point considered corresponds to an incoming or outgoing trajectory of the n-dimensional game. It therefore applies to both.

ii) <u>The I.R.G.: Locus of Tangency Points</u> (See [4]). In our case, there is no such thing as singular controls, and the theorem applies.

We look, therefore, for the locus of the points where a primary is tangent to $\mathcal{C}$. This will be given by a simultaneous solution of

$$P(\tau) = \frac{1}{4} F^2\tau^3 - F(\eta-w)\tau^2 + \left[(\eta-w)^2 + \xi^2 - F\ell\right]\tau + 2\ell(\eta-w) = 0$$

$$P'(\tau) = \frac{3}{4} F^2\tau^2 + 2F(\eta-w)\tau + (\eta-w)^2 + \xi^2 - F\ell = 0$$

The algebraic condition for this two polynomials to have a common root is that their resolvant be zero. This gives, after some calculations:

$$\Delta = F^2\xi^2\left[\left(\xi^2 + \zeta^2 - \frac{3}{2} F\ell\right)^2 + 8F\ell\zeta^2\right] - \frac{1}{4} F^4\ell^2(\zeta^2-3\xi^2 + 4F\ell) = 0$$

where $\zeta = \eta-w$. This equation is of second degree in ζ^2, and could thus be solved for ζ in terms of ξ. Using, then, the suitable root of $P'(\tau) = 0$ would give the desired locus.

But this technique gives very complicated formulas we prefer to use a different one: Take the common root τ_c as the independent parameter. The simple operation

$$P(\tau) - \tau P'(\tau) = 0$$

immediately gives η, and placing it back in $P'(\tau) = 0$ we find ξ:

$$\eta-w = \frac{\frac{1}{2} F^2\tau_c^3}{F\tau_c^2 + 2\ell}$$

$$\xi^2 = F\ell^2 \frac{F\tau_c^2 + 4\ell}{(F\tau_c^2 + 2\ell)^2} \qquad (6.10)$$

and it can be seen by direct substitution that these formulas give $\Delta = 0$. For $\tau_c = 0$, equations (6.10) give

$$\eta = w \qquad \xi = F\ell$$

which are the coordinates of B. The curve $\mathfrak{D}$ is the constrained trajectory corresponding to $\tau_c = 0$.

The locus of tangency points lying between $\mathfrak{D}$ and the B.U.P., who are tangent at B, is therefore tangent to the B.U.P. It turns out to be extremely close to $\mathfrak{D}$ and numerically very difficult to separate from it on part of its length.

It is clear that at $\tau_c = 0$, formulas (6.10) give

$$\frac{d\eta}{d\xi} = 0$$

If we notice that the B.U.P. $\mathfrak{B}$ is given by η = constant, this shows that the locus of capture points of the tangent trajectories is also tangent to $\mathfrak{B}$ at B.

For $p \geq 1$, let τ_1 be such that $Q(\tau_1) = 0$. Clearly, the tangency point corresponding to $\tau_c = \tau_1$ must be at the point of intersection of the line $\mathcal{Q}$ with $\mathcal{C}$. For $p = 1$ the locus of tangency points is tangent to the symmetry plane, and it intersects it for $p > 1$. However, constrained trajectories arriving at a point of this locus with $\tau_c > \tau_1$ must be discarded because the corresponding primary would go through $\mathcal{Q}$ before capture occurs, and thus would not be optimal.

iii) <u>The I.R.G.: Constrained Trajectories</u>. We already know that the safe contact condition gives ψ independently of φ as

$$\cos(\hat{\psi} - \theta) = \frac{v}{w} \cos\theta$$

$$\sin(\hat{\psi} - \theta) = \frac{1}{w}\sqrt{w^2 - v^2\cos^2\theta}$$

and that the corresponding dynamics are equations (5.7). The Hamiltonian is

$$\hat{H} = \frac{\lambda\theta}{\ell}\left[\sqrt{w^2 - v^2\cos^2\theta} + v\sin\theta\right] - F\left[\frac{\lambda\theta}{v}\sin\varphi - \lambda_v\cos\varphi\right] + 1$$

The optimal control is thus

$$\sin \varphi^{o} = \frac{\lambda_\theta}{v\sigma} \qquad \cos \varphi^{o} = - \frac{\lambda_v}{\sigma} \qquad \sigma = \sqrt{\frac{\lambda_\theta^2}{v^2} + \lambda_v^2}$$

which is consistent with our claim that $\varphi^{o} = \varphi$ at every point. The corresponding Hamiltonian is

$$\hat{H}^{o} = \frac{\lambda_\theta}{\ell}\left(v \sin \theta + \sqrt{w^2 - v^2 \cos^2 \theta}\right) - F\sigma + 1 = 0$$

and the path and adjoint equations for the constrained game

$$\dot{\theta} = \frac{1}{\ell}\left(v \sin \theta + \sqrt{w^2 - v^2 \cos^2 \theta}\right) - F \frac{\lambda_\theta}{v^2 \sigma}$$

$$\dot{v} = - F \frac{\lambda_v}{\sigma}$$

$$\dot{\lambda}_\theta = - \lambda_\theta \frac{v \cos \theta}{\ell} \; \frac{v \sin \theta + \sqrt{w^2 - v^2 \cos^2 \theta}}{\sqrt{w^2 - v^2 \cos^2 \theta}}$$

$$\dot{\lambda}_v = -\lambda_\theta \frac{\sin \theta}{\ell} \frac{1}{\sqrt{w^2 - v^2 \cos^2 \theta}}\left[v \sin \theta + \sqrt{w^2 - v^2 \cos^2 \theta} - v\right] + F \frac{\lambda_\theta^2}{v^3 \sigma}$$

Finally, comparing this Hamiltonian with that of the three dimensional game, we obtain

$$\frac{\lambda_\theta}{\ell} \sqrt{w^2 - v^2 \cos^2 \theta} = w\rho - v\lambda_r \cos \theta \qquad \rho = \sqrt{\lambda_r^2 + \frac{\lambda_\theta^2}{2}}$$

which can be rearranged so as to yield a perfect square that gives

$$\lambda_r = \frac{\lambda_\theta}{\ell} \frac{v \cos \theta}{\sqrt{w^2 - v^2 \cos^2 \theta}} \qquad \rho = \frac{\lambda_\theta}{\ell} \frac{w}{\sqrt{w^2 - v^2 \cos^2 \theta}}$$

We recognize the relations we had along $\mathfrak{D}$. But here ρ is no longer arbitrary, and integration of the adjoint equations is needed to provide λ_ρ and λ_v. Their initial conditions are obtained by placing (6.10) in (6.7).

The constrained trajectories actually lie between the curve of tangency points and $\mathfrak{D}$. They go around the end of $\mathfrak{D}$ in a state constrained form of Isaacs' swerve maneuver. Due to the fact mentioned earlier, the resulting two-dimensional field does not always fill the empty region of the capture cylinder. Figure 12 is a scale drawing of the capture cylinder surface, for $p = 1$.

The corresponding three dimensional field of extremals, if it does not fill the whole void left by the primaries, still gives a smooth extension of the previous field. It contains trajectories coming from the external side of $\mathcal{E}$.

For $p \geq p_2$, the curve $\mathfrak{D}$ closes, and this construction succeeds in filling the portion of the capture region non accounted for by the primaries interrupted at $\mathcal{Q}$. Therefore, it gives the complete solution of the game.

6.4 Corner Condition

For $1 \leq p \leq p_2$, we expect that for starting points with small v, the trajectories of the game of degree, similar to those of the game of kind, will first go away from the capture cylinder, and then reach the field already known and follow it. Hence the need for a corner condition.

Terminology. We are interested in the conditions that must hold on a surface S where two fields of extremals join. S locally divides the space in two regions. Let the incoming trajectories be in region −, and the outgoing ones in region +. The corresponding sensitivity vectors are λ^- and λ^*. Let n be the normal to S pointing in region +, so that by definition:

$$\langle n, f(\varphi^-, \psi^-)\rangle \geq 0$$

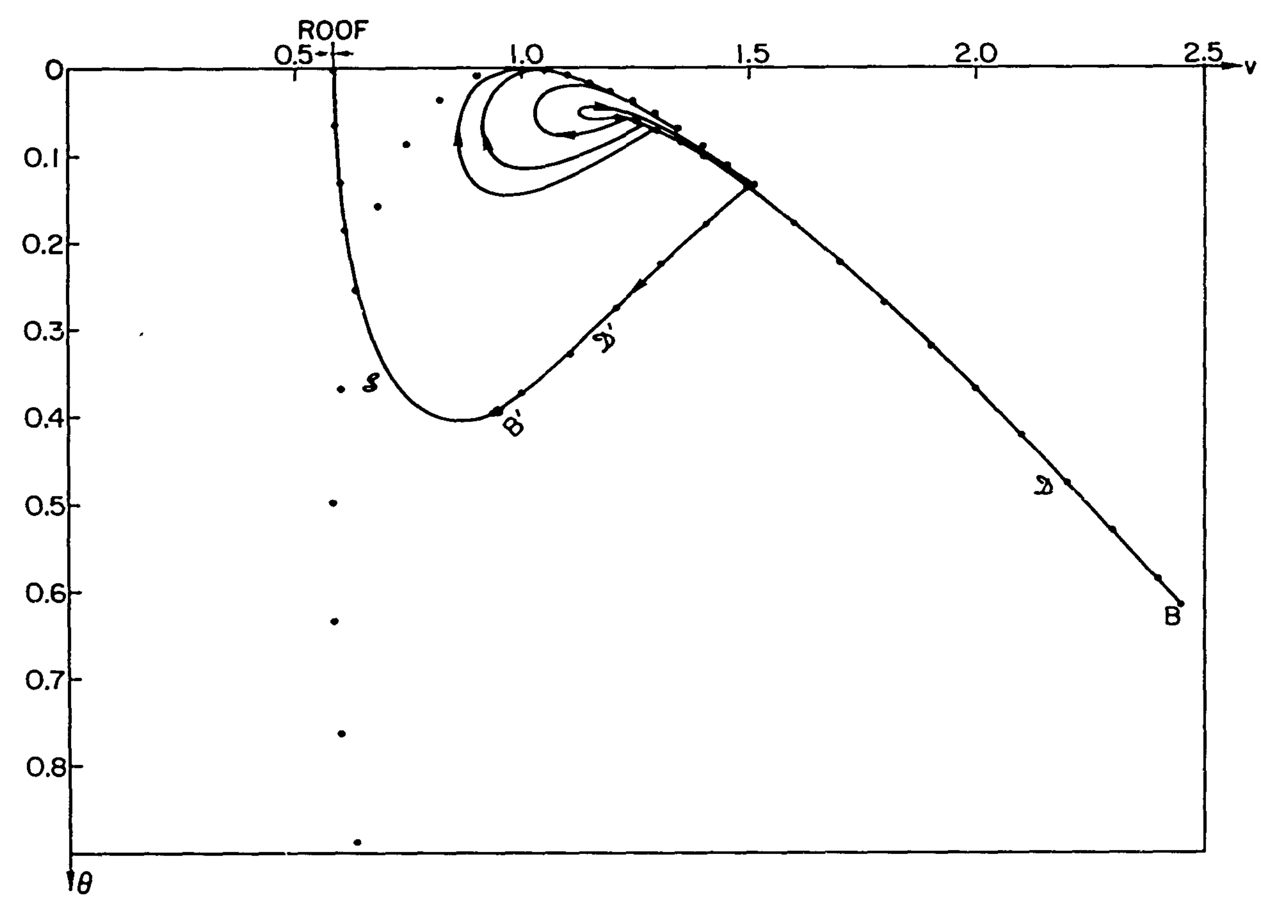

FIGURE 12. The Constrained Game p = 1

$$\langle n,\ f(\varphi^*, \psi^*)\rangle \geq 0$$

A simple result is the following "jump condition":

Proposition

$$\lambda^- = \lambda^+ + \alpha n \tag{6.11}$$

Proof. $V(z)$ is assumed to be uniquely defined in the whole space. From its values on S, its directional derivatives in a plane tangent to S are uniquely determined. And these are the projection of λ in that plane. This proves the proposition.

Notice that the argument depends on the existence of the directional derivatives of V in S, and thus on the existence of an open neighborhood of z in S. It does not hold as such on the boundary of S.

As a consequence of this proposition, once S, and thus n, is known, we can determine λ^-, the main equation for region - allowing us to calculate α. This is equivalent to saying that we can solve the game with terminal surface S and terminal payoff $V(z)|_S$ added to the integral part of the performance index. We still need a condition to determine S.

Assumption. Assume that in a neighborhood of S we have:

$$\begin{aligned} &\langle n,\ f(\varphi^-, \psi^+)\rangle \geq 0 \\ &\langle n,\ f(\varphi^+, \psi^-)\rangle \geq 0 \end{aligned} \tag{6.12}$$

Namely, none of the players can prevent the state from crossing S by keeping his optimal strategy of region$^-$. Then, we have a result similar to the classical corner condition. (See [5] and [9]).

<u>Theorem</u>. Under condition (6.12), the corner condition is

$$\lambda^- = \lambda^+ \tag{6.13}$$

<u>Proof</u>. Assume one of the players decide to switch earlier, on a surface translated from S by $\delta\ell$. Then, because of (6.12) the other player is obliged to switch also. Taking (6.11) into account, the change in payoff, to first order, is

$$\delta V = \langle(\lambda^+ - \lambda^-), \delta\ell\rangle = -\alpha\langle n, \delta\ell\rangle$$

If α is not zero, this quantity has the sign of α, and thus there exists a small enough $\delta\ell$ such that the variation in payoff has that sign. Therefore, the player whose advantage it is should have switched earlier. This ends the proof.

At this point, a new, typically game-theoretic, phenomenon occurs: assume that one of the two inequalities (6.12) does not hold. Then the player who can prevent the state from crossing S is also able to prevent his opponent from taking advantage of the potential variation of payoff we just pointed out. Therefore, a switching surface S can occur with

$$\langle n, f(\varphi^+, \psi^-)\rangle < 0 \qquad \alpha < 0 \tag{6.14a}$$

or

$$\langle n, f(\varphi^-, \psi^+)\rangle < 0 \qquad \alpha > 0 \tag{6.14b}$$

Before we investigate what new relation replaces (6.13), we make the following remark, concerning a game with separated dynamics:

<u>Proposition</u>. If the dynamics are separated, one at most of the two inequalities (6.12) can be false.

<u>Proof</u>. By simple calculation we have:

$$\langle n, f(\varphi,\psi)\rangle = \langle n,h(\psi)\rangle - \langle n, g(\varphi)\rangle$$

by definition of n, we have

$$\langle n,f(\varphi^-,\psi^-)\rangle \geq 0 \qquad \langle n, h(\psi^-)\rangle \geq \langle n, g(\varphi^-)\rangle$$

$$\langle n,f(\varphi^+,\psi^+)\rangle \geq 0 \qquad \langle n, h(\psi^+)\rangle \geq \langle n, g(\varphi^+)\rangle$$

Assume that

$$\langle n,f(\varphi^+,\psi-)\rangle < 0 \qquad \langle n,h(\psi^-)\rangle < \langle n, g(\varphi^+)\rangle$$

together with the previous two inequalities yields

$$\langle n, f(\varphi^-,\psi^+)\rangle = \langle n, h(\psi^+)\rangle - \langle n\ g(\varphi^-)\rangle > 0$$

and similarly for the other case. This proves the proposition.

The indifference condition. Assume, for definiteness, that

$$\langle n, f(\varphi^+, \psi^-)\rangle < 0 \qquad (6.14a)$$

Then, according to what we said, we can have, on S, $\alpha < 0$.

If upon reaching S the pursuer changes his control but the evader does not, the state drifts back into region −. The pursuer must switch back to φ^-, causing the state to reach S again. The sequence is then repeated, inducing "chatter". We assume a convex vectorgram so that chatter can always be replaced by, and cannot be better than, a simple strategy. Thus the pursuer must choose a control $\tilde{\varphi}$ such that the state follows S:

$$\langle n, f(\tilde{\varphi},\psi)\rangle = 0$$

we call $(\tilde{\varphi},\psi^-)$ the traversing strategies and (φ^+,ψ^+) the penetrating strategies. S can be the switching surface of the optimal game only if

the traversing strategies still optimal, namely if, as on an optimal trajectory, the rate of decrease of V is equal to the value of the Lagrangian with the corresponding controls. Notice that because of (6.11)

$$\langle \lambda^-, f(\tilde{\varphi},\psi^-)\rangle = \langle \lambda^+, f(\tilde{\varphi},\psi^-)\rangle$$

so that the Hamiltonian is uniquely defined and our condition can be written

$$H(\lambda^-,\tilde{\varphi},\psi^-) = L(\tilde{\varphi},\psi^-) + \langle \lambda^-, f(\tilde{\varphi},\psi^-)\rangle = 0$$

But we know that

$$\min_{\varphi} H(\lambda^-,\varphi,\psi^-) = H(\lambda^-,\varphi^-,\psi^-) = 0 \quad .$$

We therefore have this complement to the previous theorem.

Theorem: (The Indifference Condition). If condition (6.12) does not hold, but instead (6.14a) for instance, then at a corner point we must have $\alpha < 0$ and in addition one of the following two conditions must hold.

- $\tilde{\varphi} = \varphi^-$

or

- $\text{Arg} \min_{\varphi} H(\lambda^-,\varphi,\psi^-)$ non-unique.

The first possibility corresponds to the switch envelope, a phenomenon first discovered by J. V. Breakwell and A. W. Merz in Isaacs' homicidal chauffeur game [20]. The trajectories reach S tangentially, and then, to the choice of the evader (or of the pursuer for the case (6.14b)) the state can either follow the switch envelope, or leave it on a trajectory of region +.

The second possibility has an interesting interpretation when the Lagrangian is independent of φ, in the minimum time problem for instance.

Then P (Q in the case (6.14b)) must have an affine set as part of its boundary, and λ^- must be normal to it. In particular, if P lies in an affine set $\mathfrak{F}$ but has only relative extremal points in its boundary, (a line segment, a conic section) then λ^- must be normal to $\mathfrak{F}$.

For a two dimensional game, the only proper subspace is a straight line, so that the possibility we have just discussed can occur only if one of the players has a linear vectogram, and we recognize Isaacs' equivocal phenomenon. We see that in the higher dimensions, this is not required. In the isotropic rocket game, for instance, an equivocal surface of this type could occur with λ^- parallel to $\vec{r}$, and thus normal to the plane of P.

Let us finally notice that our description (Chapter 5) of how two semipermeable surfaces can join is the limiting case of a switch envelope. The unicity assumption we had to make merely ruled out an equivocal junction. There does not seem to be any reason why this could not occur. We did not describe it because we did not need it, and because it is clear how the present more general theory would apply. The case (6.13) leads, for barriers, to a smooth surface, since the adjoint is normal to it.

6.5 The Switch Envelope

i) The Indifference Condition. In our game, we already know a line where a corner occurs: The envelope junction. If we regard the barriers as limits of the trajectories of the game of degree, we conclude that we must find a switch envelope passing through that line. The choice of strategies, traversing or penetrating, would rest with the evader, and α be negative.

Let the switching surface be a surface X given as

$$v = v(x,y)$$

Then, its normal is

$$n = \begin{cases} -\dfrac{\partial v}{\partial x} = -p \\[2ex] -\dfrac{\partial v}{\partial y} = -q \\[2ex] 1 \end{cases}$$

(We have used the traditional notation p here, not to be confused with the parameter of our game. This notation appears only in the present subsection). According to (6.11)

$$\begin{aligned} \lambda_x^- &= \lambda_x^+ - \alpha p \\ \lambda_y^- &= \lambda_y^+ - \alpha q \\ \lambda_v^- &= \lambda_v^+ + \alpha \end{aligned} \tag{6.11a}$$

and we have two relations. The main equation of region - and the indifference condition which, here, is a tangency condition

$$H(\lambda^-,\varphi^-,\psi^-) = -F\sigma^- + w\rho^- - v\lambda_y^- + 1 = 0$$

$$-\langle n,\ f(\varphi^-,\psi^-)\rangle = -F\,\frac{\lambda_\theta^-}{v^2\sigma^-}\,(py-qx) \tag{6.15}$$

$$+\ \frac{w}{\rho^-}\left[p\lambda_x^+ + q\lambda_y^+ - \alpha(p^2 + q^2)\right] - vq + F\,\frac{\lambda_v^-}{\sigma^-} = 0$$

where

$$\rho^- = \sqrt{\rho^{+2} - 2\alpha(p\lambda_x^+ + q\lambda_y^+) + \alpha^2(p^2 + q^2)}$$

$$\lambda_\theta^- = \lambda_\theta^+ - \alpha(py - qx)$$

$$\sigma^- = \sqrt{\sigma^{+2} - 2\alpha\left(\lambda_\theta^+\,\frac{py-qx}{v^2} - \lambda_v^+\right) + \alpha^2\left[\frac{(py-qx)^2}{v^2} + 1\right]}$$

Conceptually, the parameter α can be eliminated between the two equations (6.15), leaving a first order partial differential equation on $v(x,y)$.

Actually it can be seen that this elimination can be avoided, by directly finding the equations of the characteristics of this system of equations. If the two relations are written

$$H(x,y,v,p,q,\alpha) = 0$$

$$G(x,y,v,p,q,\alpha) = 0$$

define

$$[X] = \frac{\partial H}{\partial x}\frac{\partial G}{\partial \alpha} - \frac{\partial G}{\partial x}\frac{\partial H}{\partial \alpha}$$

and similarly for [Y], [V], [P] and [Q], and then the characteristics have the same equations as for a single partial differential equation:

$$\frac{dx}{[P]} = \frac{dy}{[Q]} = \frac{dv}{p[P] + q[Q]} = \frac{-dp}{[x] + p[V]} = \frac{-dq}{[Y] + q[V]} = d\xi$$

and an additional relation is needed to propagate α. It is easy to derive as

$$d\alpha = \left[\left(\frac{\partial H}{\partial x}\frac{\partial G}{\partial p} - \frac{\partial G}{\partial x}\frac{\partial H}{\partial p}\right) + \left(\frac{\partial H}{\partial y}\frac{\partial G}{\partial q} - \frac{\partial G}{\partial y}\frac{\partial H}{\partial q}\right) + p\left(\frac{\partial H}{\partial v}\frac{\partial G}{\partial p} - \frac{\partial G}{\partial v}\frac{\partial H}{\partial p}\right)\right.$$
$$\left. + q\left(\frac{\partial H}{\partial v}\frac{\partial G}{\partial q} - \frac{\partial G}{\partial v}\frac{\partial H}{\partial q}\right)\right] d\xi$$

We now have a problem of Cauchy: pass an integral of (6.15) through $\mathcal{J}$. It is known that unless $\mathcal{J}$ is a characteristic, this problem has a well defined solution.

ii) <u>Shape of $\mathcal{K}$</u>. Two major difficulties occur in trying to actually compute an integral of (6.15).

First, on $\mathcal{J}$, λ^+ is infinite, so that only its direction is known. This is, in principle, compensated for by the fact that we have another piece of information: we know the direction of λ^-, namely ν^-. Thus we can choose the direction of n, normal to $\mathcal{J}$, in such a way that the limit of λ^- for points tending to $\mathcal{J}$ be ν^-. But this program is exceedingly difficult to carry out. It would have to be done by iteration on the position of n as a function of the point of $\mathcal{J}$ considered.

In addition, the problem is made even more difficult by the fact that λ^+ is not known explicitly as a function of the state. It is computed by completely separate means, together with the state along a trajectory. Thus, what is known numerically only, is a family of functions $z(\tau_c, \tau_1, \tau_2)$, $\lambda(\tau_c, \tau_1, \tau_2)$ where τ_c is the tangency τ, τ_1 and τ_2 are the parameters along the state constrained and the unconstrained trajectories. A computation in the (τ_c, τ_1, τ_2) space makes the quantities p and q become complicated functions of the variables that we cannot express explicitly either.

For these reasons, numerical integration of our partial differential equation did not appear to be feasible within the scope of this dissertation. But some interesting results can be found about the shape of a solution.

The way the problem of Cauchy is solved is the following: at each point of $\mathcal{J}$ the partial differential equation defines a cone of possible normals to the surface. The requirement that it be normal to $\mathcal{J}$ determines this normal. Then the necessary initial conditions are known to integrate the equations of the characteristics. At the end J of $\mathcal{J}$, the second requirement disappears, and all the directions satisfying the equations can be used. Consequently, the solution of (6.15) has the desired shape, providing a field of extremals similar to the singular surface for those arriving in the vicinity of J.

We can also show that this solution actually lies in the region we conjecture, between the envelope barrier and the roof. Equations (6.11a) are identically satisfied along a solution. They express the fact that the projections of λ^+ and λ^- on the tangent plane to $\mathcal{K}$ are equal. Therefore, by continuity, the projections of ν^+ and ν^- on this plane

at $\mathcal{J}$ must coincide. The vectors ν being normal to the two barriers, this is possible only if the tangent plane to K lies in the dihedron defining the capture region.

Finally, notice also that the directions of the vectors ν imply, as must be, $\alpha < 0$.

Figure 13 is a schematical cross section of the barriers by a plane v = constant.

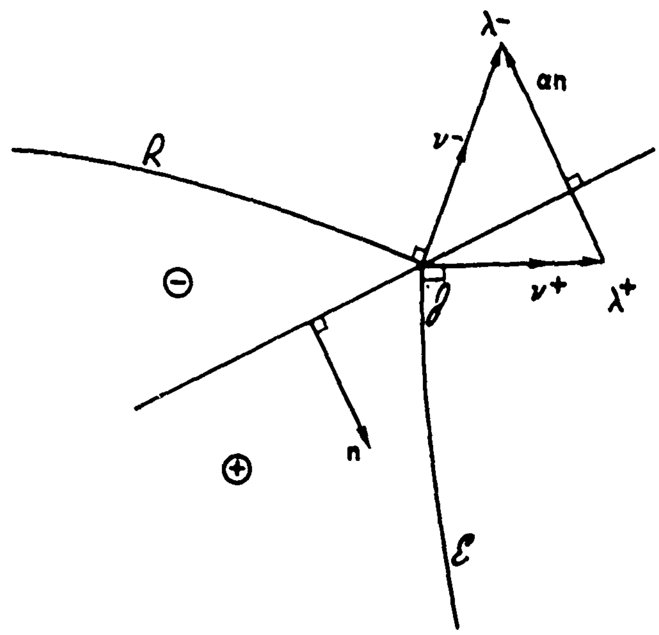

FIGURE 13. Position of the Switch Envelope

Because we have not been able to compute this surface--and consequently not the field--either, we cannot carry further the solution of this game.

Some questions arise about the possibility that the construction proposed is part of the solution of the game of degree. We shall try to answer them and to present a reasonable set of conjectures.

6.6 Conjectures

A first question arises about the trajectories of the (y,v) plane. There, we know that the optimal trajectories are a set of parabolas,

deduced from each other by translation parallel to the y-axis. But this poses two problems: These trajectories belong to the field of primaries, and we now pretend that they are imbedded in another field, making a corner with the first one. And furthermore, this corner occurs on a switch envelope, so that these parabolas should be tangent to $\mathcal{K}$. But it is obvious they have no envelope, so that they cannot be tangent to $\mathcal{K}$, unless this surface itself is tangent to the symmetry plane. This last possibility is ruled out by the fact that $\mathcal{J}$ belongs to $\mathcal{K}$, and, except for $p = 1$, is not tangent to the symmetry plane.

We conjecture the following answer to these questions. The switch envelope surface $\mathcal{K}$ intersects the (y,v) plane at a nonzero angle, and not $\pi/2$ either, along a line $\mathcal{L}$, passing through A'. The trajectories of the field - arriving tangent to $\mathcal{K}$ at $\mathcal{L}$ come from the region $x > 0$, as the trajectories of the envelope roof indicate, thus leaving between them and the symmetry plane a region not accounted for by this field.

Along $\mathcal{L}$, the field of incoming trajectories is interrupted. In other words, due to the discontinuity of the normal of $\mathcal{K}$, the restriction of $V(z)$ to $\mathcal{K}$ is differentiable in a closed half surface only. Thus, the argument that the component of λ in the tangent plane to $\mathcal{K}$ is continuous does not hold any longer. What does hold is that the component tangent to $\mathcal{L}$ is continuous. This gives one less condition.

On the other hand, for a trajectory arriving at $\mathcal{L}$ the evader cannot prevent the state from crossing $\mathcal{K}$, again because of the angle in that surface at that point. Therefore, the requirement that this trajectory be tangent to $\mathcal{K}$ does not persist either. Therefore, using the degree of freedom left in determining λ^-, we can, from each point of $\mathcal{L}$, generate a one-parameter family of extremals extending from the one tangent to $\mathcal{K}$ to the parabola. This fills the void left by the previous field. This singular roof now appears as the natural limit of this construction, giving strength to the conjecture.

Notice that the fact that $\lambda^- - \lambda^+$ must be normal to $\mathcal{L}$ and that λ^- must generate the parabolas as optimal trajectories still does not

allow to compute $\mathfrak{L}$ independently of $\mathcal{K}$, as any adjoint λ^- contained in the (y,v) plane generates the same trajectories. But we conjecture that this line lies in the region $y < v\tau_1$, $Q(\tau_1) = 0$, so that the field of primaries is interrupted between its singularity and capture. The parabolas, then go through $\mathcal{A}$, but are imbedded in a field with no singularity there.

We shall now present a conjecture giving a tentative solution of the problem of the termination of the singular carrier, left unsolved in the previous chapter. We have seen that there is a point B' where a trajectory of the singular surface $\mathcal{S}$ is tangent to the capture cylinder. Extending from B' into the capture region, there is a locus of points where the trajectories incoming to $\mathcal{K}$ are tangent to $\mathcal{C}$. From each of these points, we can generate retrogressively a constrained trajectory, and incoming to these a three dimensional field of tributaries. Qualitatively, the constrained trajectories look like $\mathfrak{D}'$ and the tributaries like the trajectories of $\mathfrak{E}'$.

Let C be the point where the trajectory of $\mathcal{S}$ through B' is tangent to the cusp on $\mathcal{S}$. From this point, the envelope barrier $\mathfrak{E}'$ cuts the trajectories of $\mathcal{K}$. Similarly, the field of tributaries just described will interfer with the field of trajectories incoming to $\mathcal{K}$. Thus, a dispersal surface will occur, a locus of points where the time to go in both fields is equal.

The trajectory of $\mathcal{S}$ through B' belongs to both fields, and C will thus belong to the dispersal surface. Therefore, this surface intersects $\mathcal{S}$, and, as the rest of its field, $\mathcal{S}$ must be interrupted along that line.

Here again, V(z) must be uniquely defined on the dispersal surface. The only way in which the field on one side can have an infinite gradient and not the field on the other side, is for the dispersal surface to be tangent to the barrier at their intersection. Figure 14 sketches this situation.

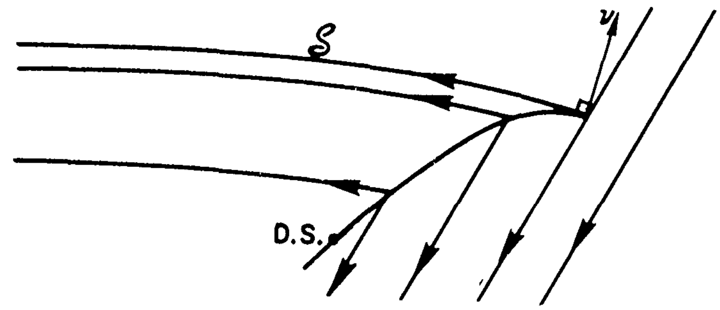

FIGURE 14. Singular Barrier and Dispersal Surface

Since following $\mathcal{S}$ does not enable the evader to avoid capture, there is no contradiction in having an optimal trajectory of the game of degree have a common point with the barrier. This is similar to what happens in the Homicidal Chauffeur [20] and other games.

This construction still does not fill the whole state space with extremals. Another field would join on the one just constructed, possibly through an equivocal surface generated from the end of the singular barrier. The common trajectory to $\mathcal{S}$ and $\mathcal{E}'$, through B' and C, also would have to be considered as it is the end of the barrier $\mathcal{E}'$, and the other edge of the "hole" left in the proposed barrier.

6.7 Conclusion

Our investigation of the Isotropic Rocket Game can be summarized as follows:

The linear theory shows that for $p < 1$, capture always occurs, in a line not larger than an estimating function we were able to construct explicitly. However, this time is not always optimal.

A more classical approach led to the following situation:

$p < 1$. We exhibited a composite field of extremals made of part of the primaries, a safe contact, and its tributaries. Although no analytic proof is available, our computations indicate that this field probably fills the space, with no conjugate point, and thus provides the complete solution to the game.

$\underline{1 \leq p < p_1}$. We have been stopped in our investigation by an excessive technical difficulty in the integration of a partial differential equation. Since any sufficiency argument rests upon the construction of a complete field of extremals, we can only propose conjectures. We have not been able to display a closed set of barriers. Furthermore, we have proposed a construction that would give trajectories going around the known barriers. Consequently, our tentative conclusion is that capture occurs from any initial condition. For outer regions of the state space, an optimal chase would then involve eight stages or more.

$\underline{p_1 \leq p \leq p_2}$. We have found a set of semipermeable surfaces forming a closed barrier: it defines a closed capture region. If the game starts outside this region capture will never occur. If it starts inside, capture will always occur. For a small portion of this region, the optimal trajectories seem to involve the same singularity as in the case $1 \leq p < p_1$, that we have not been able to compute.

$\underline{p_2 \leq P}$. We still have a closed barrier, of simpler configuration than in the previous case. The same field as in the case $p < 1$ seems to account for the whole capture region, again providing the complete solution.

Conclusions

Information Structures. We have seen that Pontryagin's direct method can be extended to various information structures. In the basic form, the pursuer knows the evader's control for a time ϵ in the future. Letting ϵ grow to infinity gives the case where the whole future control of the evader is known. Giving a fixed finite value to ϵ leads to a discrete estimating function $T(z,\epsilon)$. It is possible, in particular, to check directly that for a given point z, $T(z,\epsilon)$ increases as ϵ is decreased. Pontryagin proposes a construction, the alternating integral, that gives an estimating function valid for every positive ϵ.

This suggests an investigation of the limit of the pursuit process proposed as ϵ goes to zero.

We modify somewhat Pontryagin's process by considering the minorant game or lower ϵ-strategy where the players choose their controls sequentially. Then, using results of Fleming and of Friedman on the convergence of the game as ϵ goes to zero, we have a well-defined notion of continuous strategy that lends itself to the following analysis. We have given an explicity characterization of the strategies obtained by this variation of Pontryagin's technique. This allowed us to show that they have a well defined limit as ϵ goes to zero, that we were able to characterize in a way reminiscent of the Maximum Principle. An interesting result is that for a wide variety of games, this limit strategy of the pursuer is independent of the control chosen by the evader at current time.

Optimality. Since in the unsymmetric approach taken, the pursuer has some knowledge of the future control of his opponent, it is possible to seek a pursuit strategy optimal against that control, which is more than usual saddle point condition. The strategy proposed by Pontryagin achieves this goal provided that the estimating function is the optimal time to go in the usual sense. We therefore investigated the optimality of this pursuit time. We found a condition under which there is an ϵ small enough so that this time is locally optimal with the corresponding ϵ-strategy. Then, using our definition of a continuous strategy, we proved that for that limit, the estimating function always has this local property. Some of the intermediate results derived for these two proofs are of interest by themselves.

We have described the phenomenon by which the global trajectories can still fail to be optimal. Understanding this mechanism gives us direct means of checking the optimality of the trajectories computed in any instance. We also derived a condition under which we are assured of this optimality. Unfortunately, it is not explicit and not much simpler than the direct check. However, it allowed us to derive several more restrictive sufficient conditions, including an earlier one by Gusyatnikov and Nikolsky.

Nonregular Points. The previous analysis emphasized the importance of Pshenichnyi's nonregular points. We proved that they are points where the gradient of the estimating function is infinite. We distinguished between two kinds of such points. The first kind was shown to be associated with Isaacs' concept of a barrier. The second kind is not as well understood. An explicit example was nevertheless given to show that it can occur.

Multistage Games. The construction proposed by Pontryagin naturally leads to the consideration of multistage games. It was seen that most of the concepts introduced in the continuous case carry over to the discrete case. Our main interest has been in a slight modification of the resulting discrete theory: the system-theoretic formulation where the controls are unbounded and capture is defined as point coincidence modulo a subspace. In that case, the constructions performed take a very simple form, and explicit criteria were given on the coefficients of the matrices involved for capture to be possible with each of the three main information structures. We also considered the case where the capture subspace is invariant under the free dynamics of the system, and generalized to the vector valued controls case an earlier strong controllability theorem by Kalman.

Isotropic Rocket Game. In the second part, we investigated an example considered by Isaacs: the Isotropic Rocket Game. An object having an acceleration of constant magnitude tries to get within a distance ℓ of an object having a velocity of constant magnitude. We generalized Isaacs' formulation by assuming that the chase occurs in the three dimensional physical space, but immediately proved that an optimal chase occurs in a fixed plane, and has, in a suitable set of reference axes, linear dynamics. The state space is then four dimensional, as the components of the pursuer's velocity must be considered together with the relative coordinates of the evader with respect to the pursuer. The linear theory was applied to that representation. It showed, in particular, that when the unique parameter p of the game is smaller than one, capture occurs from all initial conditions. This is a good approximation of the complete situation in that the limiting value of the parameter we found is very close to one. However, this game exhibits the typical phenomenon

where the linear theory fails to yield the optimal solution. Therefore we could not conclude from that theory that for $p > 1$ the evader would escape, and we found later that this is probably not true.

<u>Game of Kind</u>. We used a reduced three dimensional state space to pursue the analysis of this game using Isaacs' generalization of the Hamilton Jacobi theory. We first investigated the game of kind. We emphasized the concept of local cone of semipermeable directions, and gave a geometrical construction of that cone. In particular, we were led to the consideration of a new type of semipermeable surface generated from one point by that cone of directions: the singular surface. We also used that concept to give a simple description of Isaacs' envelope barrier concept. We saw that this envelope barrier can be considered as a particular application of a more general result as to how two semipermeable surfaces can join at a non zero angle in a barrier: the envelope junction.

<u>Game of Degree</u>. We finally investigated the game of degree, for which we proposed a derivation of the generalized Hamilton-Jacobi equation by Carathéodory's technique. The game of degree involves a safe contact first pointed out by Breakwell and Boardman. It is similar to Isaacs' envelope barrier. We introduced the concept of singular state constraint and proved that if a state constraint is not singular, optimal trajectories reach and leave it tangentially. This result can be shown to imply a result by Weierstrass about the same question. We also derived a general corner condition for differential games. It was found that it can take two forms: either the same requirement as in the one player optimization that the adjoint vector be continuous, or, if a certain inequality is not satisfied, the indifference condition. This condition itself can take two forms, one having Isaacs' equivocal phenomenon as a particular case, the other one being Breakwell's switch envelope.

The detailed solutions for the Isotropic Rocket Game was discussed at the end of Chapter Six. A detailed description of the barrier was given at the end of Chapter Five. We left the problem unsolved for a small range of the parameter p. The solution in this interesting case

seems to be very complicated. We conjectured that it involves at least an eight-stage chase, with two corners and two safe contacts.

Appendix A

EQUATIONS OF THE EXTREMALS

We start with the simplest, four-dimensional description of the game

$$\begin{aligned} \dot{X} &= -U + w_x \\ \dot{Y} &= -V + w_y \\ \dot{U} &= F_X \\ \dot{V} &= F_Y \end{aligned} \tag{A.1}$$

and we introduce the reduced coordinates

$$\begin{aligned} v &= \sqrt{U^2 + V^2} \\ x &= \frac{1}{v}(VX - UY) \\ y &= \frac{1}{v}(UX + VY) \quad . \end{aligned} \tag{A.2}$$

We define the angle A between the two systems by

$$\sin A = \frac{U}{v}$$

$$\cos A = \frac{V}{v}$$

and introduce new control variables φ and ψ by

$$F_X \sin A + F_Y \cos A = F \cos \varphi; \quad F_X \cos A - F_Y \sin A = F \sin \varphi ;$$

$$w_X \sin A + w_Y \cos A = w \cos \psi; \quad w_X \cos A - w_Y \sin A = F \sin \psi ;$$

so that φ and ψ are the angles defining the position of F and w in the new coordinate system, all angles being measured clockwise from the y axis.

Differentiating (A.2) and placing (A.1) in it, and taking the last transformation into account, we obtain the equations of the game in the three-dimensional representation

$$
\begin{aligned}
\dot{x} &= - F \frac{y}{v} \sin \varphi + w \sin \psi \\
\dot{y} &= F \frac{x}{v} \sin \varphi + w \cos \psi - v \\
\dot{v} &= F \cos \varphi \; ;
\end{aligned}
\tag{A.3}
$$

and we want the equations in cylindrical coordinates

$$
\begin{aligned}
x &= r \sin \theta \\
y &= r \cos \theta \; .
\end{aligned}
\tag{A.4}
$$

We differentiate (A.4), place (A.3) in it and solve for r and $\dot{\theta}$. We obtain

$$
\begin{aligned}
\dot{r} &= w \cos(\psi - \theta) - v \cos \theta \\
\dot{\theta} &= - \frac{F}{v} \sin \varphi + \frac{w}{r} \sin(\psi - \theta) + \frac{v}{r} \sin \theta \\
\dot{v} &= F \cos \varphi \; .
\end{aligned}
\tag{A.5}
$$

Now, we need the equations relating the various representations of $\nabla J = \lambda$ [we call J the value of the game to avoid confusion with the coordinate V of (A.1)]. Let J be given by

$$J = J_1(X, Y, U, V) = J_2(x, y, v).$$

Let

$$\Lambda_X = \frac{\partial J_1}{\partial X} \qquad \lambda_x = \frac{\partial J_2}{\partial x}$$

$$\Lambda_Y = \frac{\partial J_1}{\partial Y} \qquad \lambda_y = \frac{\partial J_2}{\partial y}$$

$$\Lambda_U = \frac{\partial J_1}{\partial U} \qquad \lambda_v = \frac{\partial J_2}{\partial v}$$

$$\Lambda_V = \frac{\partial J_1}{\partial V}$$

Taking partial derivatives of (A.2),

$$\frac{\partial x}{\partial U} = -\frac{1}{v}\left(Y + \frac{Ux}{v^2}\right) \qquad \frac{\partial x}{\partial V} = -\frac{1}{v}\left(\frac{Vx}{v^2} - X\right)$$

$$\frac{\partial y}{\partial U} = -\frac{1}{v}\left(\frac{Uy}{v^2} - X\right) \qquad \frac{\partial y}{\partial V} = -\frac{1}{v}\left(\frac{Vy}{v^2} - Y\right)$$

$$\frac{\partial v}{\partial U} = \frac{U}{v} \qquad \frac{\partial v}{\partial V} = \frac{V}{v}$$

Placing this and the other trivial partials of (A.2) in $\partial J_1/\partial X$, we obtain, after some rearrangement

$$\Lambda_X = \frac{\partial J_1}{\partial X} = \frac{\partial J_2}{\partial x}\frac{\partial x}{\partial X} + \frac{\partial J_2}{\partial y}\frac{\partial y}{\partial X} + \frac{\partial J_2}{\partial v}\frac{\partial v}{\partial X} = \frac{1}{v}(V\lambda_x + U\lambda_y)$$

$$\Lambda_Y = \frac{\partial J_1}{\partial Y} = \frac{\partial J_2}{\partial x}\frac{\partial x}{\partial Y} + \frac{\partial J_2}{\partial y}\frac{\partial y}{\partial Y} + \frac{\partial J_2}{\partial v}\frac{\partial v}{\partial Y} = \frac{1}{v}(-U\lambda_x + V\lambda_y)$$

from which it follows that

$$\lambda_x = \frac{1}{v}(V\Lambda_X - U\Lambda_Y) \tag{A.6}$$
$$\lambda_y = \frac{1}{v}(U\Lambda_X + V\Lambda_Y)$$

3

which simply checks that (λ_x, λ_y) and (Λ_X, Λ_Y) are the same vector $\vec{\lambda}_Z$ expressed in their respective coordinate systems. By the same means, we compute Λ_U and Λ_V

$$\Lambda_U = -\left(\frac{Ux}{v} + Y\right)\frac{\lambda x}{v} - \left(\frac{Uy}{v} - X\right)\frac{\lambda y}{v} + \frac{U}{v}\lambda_v$$

$$\Lambda_V = -\left(\frac{Vx}{v} - X\right)\frac{\lambda x}{v} - \quad \frac{Vy}{v} - Y)\frac{\lambda y}{v} + \frac{V}{v}\lambda_v$$

from which it follows that

$$\lambda_v = \frac{1}{v}(U\Lambda_U + V\Lambda_V) \tag{A.6a}$$

which means the dot product $\vec{v} \cdot \vec{\lambda}_v$ is conserved in the change of coordinates . We obtain also, after a rather lengthy calculation

$$\sigma^2 \triangleq \Lambda_U^2 + \Lambda_V^2 = \frac{1}{v^2}(y\lambda_x - x\lambda_y)^2 + \lambda_v^2$$

while (A.7)

$$\rho^2 \triangleq \Lambda_X^2 + \Lambda_Y^2 = \lambda_x^2 + \lambda_y^2$$

is obvious from (A.6).

Finally, we make a purely geometric transformation on the radius vector r and on a perpendicular axis

$$\begin{aligned} r\lambda_r &= x\lambda_x + y\lambda_y \\ \lambda_\theta &= y\lambda_x - x\lambda_y \end{aligned} \tag{A.8}$$

hence, obtaining other forms for ρ and σ

$$\begin{aligned} \rho^2 &= \lambda_r^2 + \frac{\lambda_\theta^2}{r^2} \\ \sigma^2 &= \frac{\lambda_\theta^2}{v^2} + \lambda_v^2 . \end{aligned} \tag{A.8a}$$

This justifies the notations used in the main body of this dissertation

We are going to find the equations of the extremals. The Euler-Lagrange equations are more easily integrated in the four-dimensional representation But as we know that the absolute orientation in the geometrical space is ignorable, we shall always assume that at the instant of capture, $\tau=0$, we have $A = 0$, or $U = 0$, $V = v$. Consequently, the expressions for Λ_U and Λ_V which we have are

$$\Lambda_X = \lambda_u$$

$$\Lambda_Y = \lambda_y$$

$$\Lambda_U \triangleq \mu_o = -\frac{\lambda_\theta}{v}$$

$$\Lambda_V \triangleq \lambda v_o = \lambda_v$$

and we shall use these relations in the equations of the trajectories because our ultimate aim is to have their equations in the three dimensional system, as a function of the parameters x_o, y_o, v_o, λ_{xo}, λ_{yo}, at $\tau = 0$. As the system is time invariant, the same equations can be used from any such set of "final" conditions, even if they do not correspond to capture. In particular, we have used them to compute trajectories in coming to the various singularities in the game. The Hamiltonian is

$$H = -\Lambda_X U - \Lambda_Y V + \Lambda_X w_X + \Lambda_Y w_Y + \Lambda_U F_X + \Lambda_V F_Y + 1$$

and its maximized form is

$$H^* = -\Lambda_X U - \Lambda_Y V + \rho w - F\sigma + 1 = -F\sigma + w\rho - v\lambda_y + 1 = 0 .$$

The adjoint equations are

$$\dot{\Lambda}_X = 0 \qquad \Lambda_X = \text{constant}$$

$$\dot{\Lambda}_Y = 0 \qquad \Lambda_Y = \text{constant}$$

$$\dot{\Lambda}_U = \Lambda_X \qquad \Lambda_U = -\Lambda_X \tau + \mu_o$$

$$\dot{\Lambda}_V = \Lambda_Y \qquad \Lambda_V = -\Lambda_Y \tau + \lambda_{v_o}$$

$$\sigma = \sqrt{\rho^2 \tau^2 - 2c_o \tau + \sigma_o^2}$$

where

$$\sigma_o \triangleq \sqrt{\mu_o^2 + \lambda_{v_o}^2}$$

$$c_o \triangleq \mu_o \Lambda_X + \lambda_{v_o} \Lambda_Y \,.$$

Λ_X and Λ_Y will be used to denote the final value of λ_x and λ_y. The dynamics, with the optimal controls, are, for the velocities

$$\frac{dU}{d\tau} = -\dot{U} = F\frac{\Lambda_U}{\sigma} = F\frac{-\Lambda_X \tau + \mu_o}{\sigma(\tau)}$$

$$\frac{dV}{d\tau} = -\dot{V} = F\frac{\Lambda_V}{\sigma} = F\frac{-\Lambda_Y \tau + \lambda_{v_o}}{\sigma(\tau)} \,.$$

Define $I_n(\tau)$ as

$$I_n(\tau) = \int_o^\tau \frac{\xi^n}{\sigma(\xi)} \, d\xi$$

The above equations give

$$U = -F(\Lambda_X I_1 - \mu_o I_o)$$

$$V = -F(\Lambda_Y I_1 - \lambda_{v_o} I_o) + v_o \,.$$

The other two state variables are then given by

$$X = x_o + \int_o^\tau U d\tau - \frac{\Lambda_X}{\rho} w\tau$$

$$Y = y_o + \int_o^\tau V d\tau - \frac{\Lambda_Y}{\rho} w\tau .$$

We integrate by parts

$$\int_o^\tau U d\tau = \tau U - \int_o^\tau \tau U' d\tau$$

and similarly for V we find

$$X = x_o - w\frac{\Lambda_X}{\rho}\tau + U\tau + F(\Lambda_X I_2 - \mu_o I_1)$$

$$Y = y_o - w\frac{\Lambda_Y}{\rho}\tau + V\tau + F(\Lambda_Y I_2 - \lambda_{Vo} I_1) . \tag{A.9}$$

The I_k's can be expressed analytically. Clearly, we have

$$I_o = \int_o^\tau \frac{d\xi}{\sqrt{\rho^2\xi^2 - c_o\xi + \sigma_o^2}} = \frac{1}{\rho}\log\left|\frac{\rho^2\tau + \rho\sigma - c_o}{\rho\sigma_o - c_o}\right| .$$

Then I_1 is obtained from the remark that

$$\rho^2 I_1 - c_o I_o \doteq \sigma(\tau) - \sigma_o \triangleq K(\tau)$$

and I_2 is obtained from the fact that

$$\frac{d}{d\xi}[\xi\sigma(\xi)] = 2\rho^2\frac{\xi^2}{\sigma(\xi)} - 3c_o\frac{\xi}{\sigma(\xi)} + \sigma_o^2\frac{1}{\sigma(\xi)}$$

so that

$$I_2 = \frac{1}{2\rho^2}[\sigma\tau + c_o I_1 - \sigma_o^2 I_o] .$$

The expressions for U and V can be made simpler by introducing, instead of I_o

$$L \triangleq (\lambda_{v_o}\Lambda_X - \mu_o\Lambda_Y)I_o$$

and it gives

$$U = -\frac{F}{\rho^2}[\Lambda_X K(\tau) + \Lambda_Y L(\tau)]$$
$$V = -\frac{F}{\rho^2}[\Lambda_Y K(\tau) - \Lambda_X L(\tau)] + v_o \quad \text{(A.10)}$$

x, y, and v are obtained placing (A.9) and (A.10) in (A.2), with the expressions we have given for I_o, I_1, and I_2. The adjoint are obtained using (A.6)

$$\lambda_x = \frac{1}{v}[\Lambda_X v_o + FL(\tau)]$$
$$\lambda_y = \frac{1}{v}[\Lambda_Y v_o - FK(\tau)] \quad \text{(A.11)}$$
$$\lambda_v = \frac{1}{v}\left\{\left[\frac{F}{\rho}K(\tau) - v_o\Lambda_Y\right]\tau - \frac{F}{\rho^2}\left[c_o K(\tau) + (\mu_o\Lambda_Y - \lambda_{v_o}\Lambda_X)^2 I_o\right] + \lambda_{v_o}v_o\right\}.$$

The general formulas for x, y, and v are exceedingly complicated. We have done the computation using the intermediary quantities X, Y, U, and V. This completes determination of the extremals in the general case.

However, our formulas break down for $c_o = \rho\sigma_o$, because I_o is no longer defined. This happens when the vectors (μ_o, λ_{v_o}), and (Λ_X, Λ_Y) are parallel, and in the case of the primaries in particular.

Assume we have

$$\mu_o = \frac{\sigma_o}{\rho}\Lambda_X$$
$$\lambda_{v_o} = \frac{\sigma_o}{\rho}\Lambda_Y \quad \text{(A.12)}$$

Placing this in the equations for Λ_U and Λ_V, they become

$$\Lambda_U = \Lambda_X(-\tau + \frac{\sigma_o}{\rho})$$

$$\Lambda_V = \Lambda_Y(-\tau + \frac{\sigma_o}{\rho})$$

$$\sigma = |\sigma_o - \rho\tau| \qquad K(\tau) = |\sigma_o - \rho\tau| - \sigma_o \; .$$

Then the velocities are given by

$$\frac{dU}{d\tau} = F \frac{\Lambda_X}{\rho} \frac{\lambda_o - \rho\tau}{|\lambda_o - \rho\tau|}$$

$$\frac{dV}{d\tau} = F \frac{\Lambda_Y}{\rho} \frac{\lambda_o - \rho\tau}{|\lambda_o - \rho\tau|}$$

and integrating, we find a formula valid for all τ's

$$\begin{aligned} U &= -F \frac{\Lambda_X}{\rho^2} K(\tau) \\ V &= -F \frac{\Lambda_Y}{\rho^2} K(\tau) + v_o \; . \end{aligned} \tag{A.13}$$

Then, we must consider the integrals

$$X_p = \int_0^\tau U(s)ds$$

integrating first from $\tau = 0$ to $\tau = \sigma_o/\rho$, then for $\tau > \sigma_o/\rho$

$$\tau \leq \frac{\sigma_o}{\rho} \qquad X_p = F \frac{\Lambda_X}{\rho^2} \int_0^\tau (-\rho s)ds = F\frac{\Lambda_X}{\rho} \frac{\tau^2}{2} \; ,$$

$$\tau > \frac{\sigma_o}{\rho}$$

$$X_p = - F \frac{\Lambda_X}{\rho^2}\left[\frac{\sigma_o^{\,2}}{2\rho} - \int_{\sigma_o/\rho}^{\tau} (\rho s - 2\sigma_o)\, ds\right] = F \frac{\Lambda_X}{\rho^2}\left[\rho \frac{\tau^2}{2} - 2\sigma_o \tau + \frac{\sigma_o^2}{\rho}\right]$$

We notice that these two expressions can be written as a single formula

$$X_p = F \frac{\Lambda_X}{2\rho^2}\left[(\tau - \frac{\sigma_o}{\rho})K(\tau) - \tau\sigma_o\right]$$

$$Y_p = F \frac{\Lambda_Y}{2\rho^2}\left[(\tau - \frac{\sigma_o}{\rho})K(\tau) - \tau\sigma_o\right] - v_o \tau .$$

Now, if we take the corresponding expressions for the general case, place (A.12) in them assuming that I_o is defined, we find identically the same equations. Consequently, one can use the same formulas, checking first whether $c_o - \rho\sigma \neq 0$, and assigning any value, say zero, to I_o if $c_o = \rho\sigma_o$.

Finally, since it gives simple formulas, it is worth deriving the equations for the reduced coordinates in the case of the primaries where $\lambda_{v_o} = \mu_o = 0$. Placing this in our formulas, we find

$$X = x_o - \frac{\Lambda_X}{\rho} (w + F \frac{\tau}{2})\tau$$

$$Y = y_o - \frac{\Lambda_Y}{\rho} (w + \frac{F\tau}{2})\tau + v_o \tau$$

$$U = - F \frac{\Lambda_Y}{\rho} \tau$$

$$V = - F \frac{\Lambda_X}{\rho} \tau + v_o .$$

We apply to this the transformation (A.2), and introduce the parameters

$$x_o = \ell \sin \beta \qquad \Lambda_X = \rho \sin \beta$$

$$y_o = \ell \cos \beta \qquad \Lambda_Y = \rho \cos \beta$$

$$v_o = s \qquad \lambda_{v_o} = 0$$

and the intermediary quantities

$$\xi = s \sin \beta$$

$$\eta = s \cos \beta \; ;$$

we obtain for the three-dimensional representation

$$x = \frac{\xi}{v} Q(\tau)$$

$$y = \frac{\eta - F\tau}{v} Q(\tau) + v\tau$$

$$v = [\xi^2 + (\eta - F\tau)^2]^{\frac{1}{2}}$$

where $Q(\tau) = F\tau^2/2 - w\tau + \ell.$

The adjoints are

$$\lambda_x = \rho \frac{\xi}{v}$$

$$\lambda_y = \rho \frac{\eta - F\tau}{v} \qquad \text{(A.15)}$$

$$\lambda_v = -\rho\tau \frac{\eta - F\tau}{v} .$$

It is interesting to notice some simple combinations

$$\lambda_o = \rho\xi\tau$$

$$\sigma = \rho\tau$$

and $H = 1 + (w - \eta)\rho = 0$, yielding

$$\rho = \frac{1}{\eta - w} .$$

These are all the equations we need in the last two chapters for the theory as well as for numerical computations.

Appendix B

COMPUTATION OF $\mathcal{J}$

We want to compute a curve $\mathcal{J}$ on the surface $\mathfrak{S}$. Let us review how the various elements of the problem are given.

The curve $\mathfrak{D}$ is obtained by numerical integration of Eq. (5.8). We consider it as parametrized by the value s of the velocity. Once a point of $\mathfrak{D}$ is known, we have analytical formulas for the trajectory of $\mathfrak{S}$ tangent at that point and for the adjoints along it, as a function of the time to go τ, from the point of $\mathfrak{S}$ to $\mathfrak{D}$. Therefore, a point of $\mathfrak{S}$ is represented by its parameters (s, τ_1). $\mathcal{J}$ is defined by an initial point and a field of directions on $\mathfrak{S}$. This direction is obtained, at any given point of $\mathfrak{S}$, by solving numerically Eq. (5.11a) for γ . This angle allows us to compute ν with $H_1 = 0$. This vector can be placed in the dynamics to obtain the direction $f(\nu, z)$.

We cannot simply use an integration routine with that direction f for two reasons: first, due to round-off and truncation errors, the curve would quickly drift off the surface $\mathfrak{S}$. Even if this was avoided, it would not give the parameters of the computed point on $\mathfrak{S}$. As we need these parameters to compute ν^b and subsequently γ, we would have to do some extremely tedious interpolation to find them.

For this reason, we directly look for $\mathcal{J}$ in the form $\tau_1 = \tau_1(s)$. Using s as the independent variable is convenient because the increment on s is then chosen arbitrarily, and (5.8) integrated once up to the new value. The rest of the integrating procedure is done by varying τ_1, using analytical expressions to obtain the corresponding point $z = (x, y, v)$.

We want to copy a classical integration routine of numerical analysis. Such techniques use the quantity $d\tau_1/ds$ that we do not have. But more precisely, what is really needed are such expressions as

$$\delta\tau_1 = \frac{d\tau_1}{ds}\,\delta s \tag{B.1}$$

where several such increments, with various arguments of the function $d\tau_1/ds$, are averaged in a suitable way. In our case, instead of computing $d\tau_1/ds$, we compute a vector ν normal to the intersecting surface, thus to $\mathcal{J}$. $\delta\tau_1$ is obtained by computing the value $\tau_1^o + \delta\tau_1$ of τ_1 where the trajectory of parameter $s^o + \delta s$ intersects the plane normal to ν through z^o. This is done by solving for τ_1 in

$$\vec{\nu} \cdot \delta\vec{z} = 0 \tag{B.2}$$

where

$$\delta\vec{z} = z(\tau_1, s^o + \delta s) - z^o .$$

Since we have analytical formulas for $\dot{z}(\tau_1)$, we can differentiate the dot product with respect to τ_1, and thus solve (B.2) in an efficient way with Newton's technique. Notice that for $\mathcal{J}$ the angle between the trajectory of $\mathcal{S}$ and the tangent plane to the roof is large enough so that this technique is quite accurate.

In our case, computing ν is a rather long operation. We have tried to keep the number of times we compute it as low as possible. For that reason, we have used an Adams method of order four, started with a Runge Kutta method of order four.

We give, thereafter, scale drawings of the crest (computed by the same program) and of $\mathcal{J}$ for $p = 1$ and $p = 1.001$.

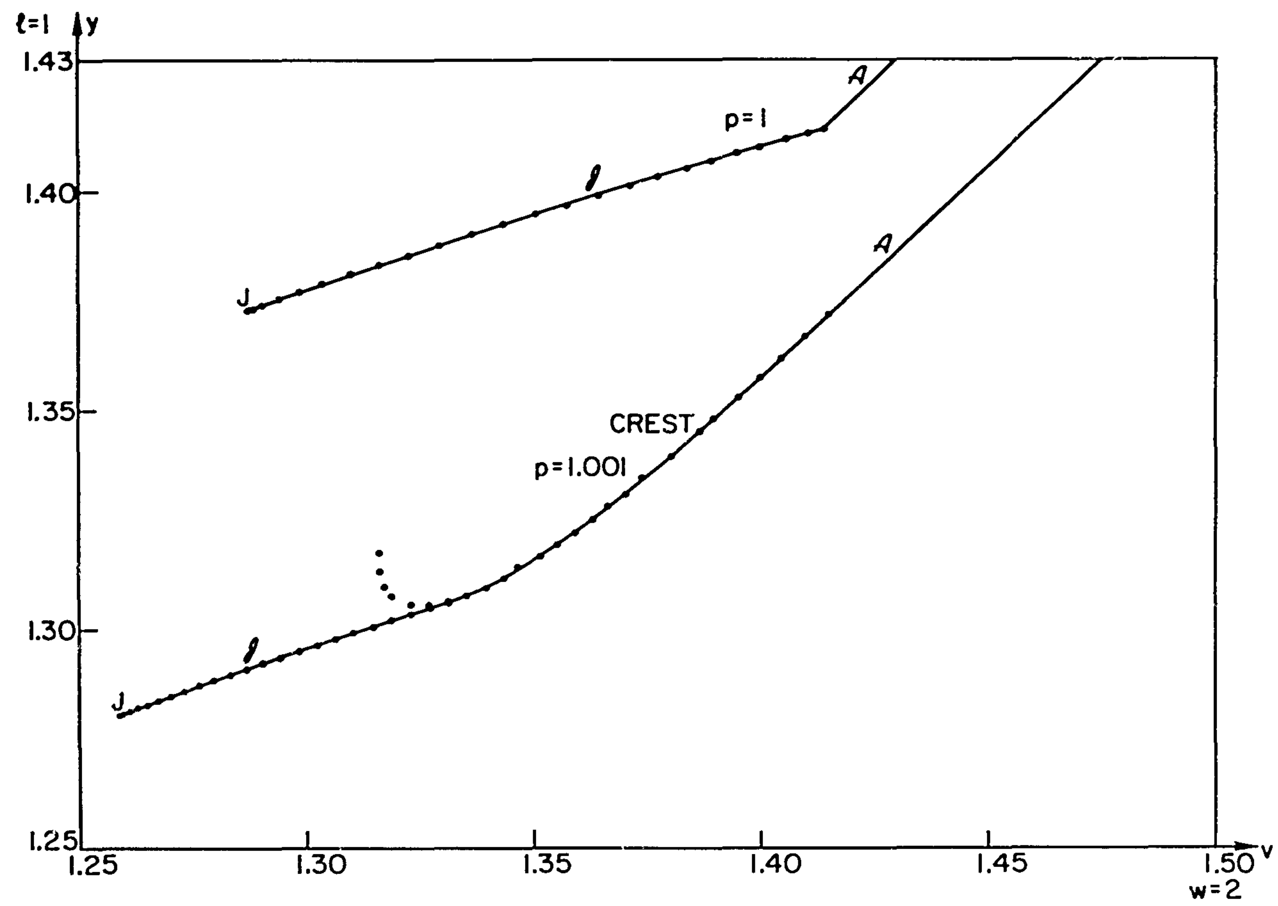

FIGURE 15. Junction $\mathcal{J}$ and Crest for p = 1 and 1.001

Appendix C

ISAACS' NOTATIONS AND OURS

Isaacs 105-116	Isaacs 244-254	This Work
$x_E - x$	-	X
$y_E - y$	-	Y
u	-	U
v	-	V
$\underline{r}$	-	$\vec{\sigma\tau}$
$\underline{U}$	-	$\vec{v}$
X	x	x
Y	y	y
v	v	v
-	θ	θ
-	ν_1	ν_x
-	ν_2	ν_y
-	ν_3	ν_v
-	U	ν_θ
ρ	ρ_1	σ
ρ_E	ρ_2	ρ
$V_{x_E} = -V_x$	-	Λ_X
$V_{y_E} = -V_y$	-	Λ_Y
V_u		Λ_U
V_v	-	Λ_V
$\sqrt{s_3^2 + s_4^2}$	s	s
s_5	-	$\rho + \beta$
$Q(\tau)$	$Q(\tau)$	$Q(\tau)$
-	a	a
-	c	c

Appendix D

REGULAR STATE CONSTRAINT AND A RESULT BY WEIERSTRASS

In Section 6.3 we defined a regular state constraint and show that if an optimal path contains such an arc, it is reached and left tangentially. This result obviously applies to the control problem, the one-player game. Similar problems were studied in calculus of variations, and it is interesting to relate the two approaches.

The problem of joining an unconstrained arc with a constrained one was considered by Weierstrass, and appeared in his lectures as early as 1879 at least [5]. The problem he deals with is his parametrical form of the calculus of variations. In control theory language, it can be stated as a problem in a two-dimensional state space with the simplest dynamics

$$\begin{aligned} \dot{x}_1 &= U_1 \\ \dot{x}_2 &= U_2 , \end{aligned} \tag{D.1}$$

and the objective is to minimize

$$J = \int_A^B F(x_1, x_2, U_1, U_2)dt$$

where F is assumed to be of class C^3 in all of its arguments (for our purposes, C^2 suffices).

Weierstrass wants the integral to depend only on the geometrical path followed. He shows that the necessary and sufficient condition for this to be true is that F be homogeneous of degree one in (U_1, U_2).

$$F(x_1, x_2, kU_1, kU_2) = kF(x_1, x_2, U_1, U_2), \quad k > 0 ,$$

and a consequence of this is that there exists a function

$$F_1(x_1, x_2, U_1, U_2)$$

such that

$$\begin{aligned} F_{U_1U_1} &= U_2^2F_1 \\ F_{U_1U_2} &= -U_1U_2F_1 \\ F_{U_2U_2} &= U_1^2F_1 \end{aligned} \tag{D.2}$$

where

$$F_{U_iU_j} = \frac{\partial^2 F}{\partial U_i \, \partial U_j}$$

$$i = 1,2$$

$$j = 1,2 \; .$$

We add that the second partial derivatives of a homogeneous function of degree one are homogeneous of degree minus one; thus, F_1 is homogeneous of degree minus three in (U_1, U_2).

We introduce Weierstrass' excess function, which, because of the homogeneity of F can be written as

$$\begin{aligned} E(x_1,x_2,U_1,U_2,\tilde{U}_1,\tilde{U}_2) = {} & \tilde{U}_1\left[F_{U1}(x_1,x_2,\tilde{U}_1,\tilde{U}_2) - F_{U1}(x_1,x_2,U_1,U_2)\right] + \\ & + \tilde{U}_2\left[F_{U2}(x_1,x_2,\tilde{U}_1,\tilde{U}_2) - F_{U2}(x_1,x_2,U_1,U_2)\right] \end{aligned}$$

assuming that the controls are the cosine and the sine of the same angle γ (we can always chose $U_1^2 + U_2^2 = 1$ by rescaling and using the homogeneity of F), and using the mean value theorem, one can express E by means of F_1 . The form found shows that if

$$F_1(x_1,\ x_2,\ \cos\gamma,\ \sin\gamma) = 0 \qquad \forall\gamma$$

then the relation E = 0 can be verified only if

$$U_1 = \tilde{U}_1 \qquad U_2 = \tilde{U}_2$$

which shows that the boundary is met and left tangentially. The problem of calculus of variations is said to be regular if (D.3) holds at every point. Notice that from the homogeneity of F_1, (D.3) is equivalent to

$$F_1(x_1,\ x_2,\ U_1,\ U_2) \neq 0 \qquad \text{if} \qquad (U_1^2 + U_2^2) \neq 0\ . \qquad \text{(D.3a)}$$

We shall assume that the coordinates have been chosen in such a way that the boundary of the state space is $x_2 = 0$. We want to compare the condition (D.3) with ours, which can be expressed, in this case, as follows: Let the optimal control, obtained by minimizing the Hamiltonian, be given by

$$U_1^* = U_1(x_1,\ x_2,\ \lambda_1,\ \lambda_2)$$

$$U_2^* = U_2(x_1,\ x_2,\ \lambda_1,\ \lambda_2)$$

and we ask that the equation

$$U_2 = 0$$

be solvable for λ_2, or in other words, that there exist a function Λ_2 such that

$$U_2[x_1,\ x_2,\ \lambda_1,\ \Lambda_2(x_1,\ x_2,\ \lambda_1)] = 0\ . \qquad \text{(D.4)}$$

We form the Hamiltonian

$$H = \lambda_1 U_1 + \lambda_2 U_2 + F(x_1, x_2, U_1, U_2).$$

As U_1 and U_2 are unrestricted, the minimum must occur at an interior point, thus given by

$$\begin{aligned} \frac{\partial H}{\partial U_1} &= \lambda_1 + F_{U1} = 0 \\ \frac{\partial H}{\partial U_2} &= \lambda_2 + F_{U2} = 0\ . \end{aligned} \tag{D.5}$$

Moreover, since we want a minimum, and not a maximum, we must have

$$\frac{\partial^2 H}{\partial U_i{}^2} = U_j^2 F_1 > 0 \qquad \begin{matrix} i = 1,2 \\ j = 1,2 \end{matrix}\ . \tag{D.5a}$$

Equation (D.5) must be used to determine U_1 and U_2, (D.5a) allowing to choose between two solutions. First, notice that multiplying the first equation in (D.5) by U_1, the second by U_2 and adding, and then making use of Euler's identity for homogeneous functions, we find

$$\lambda_1 U_1 + \lambda_2 U_2 + U_1 F_{U1} + U_2 F_{U2} = \lambda_1 U_1 + \lambda_2 U_2 + F = 0$$

showing that a consequence of (D.5) is $H = 0$. This is not necessarily possible for any pair (λ_1, λ_2).

Let us differentiate (D.5) at a fixed point (x_1, x_2), using (D.2)

$$\begin{aligned} U_2^2 F_1 dU_1 - U_1 U_2 F_1 dU_2 &= -d\lambda_1 \\ -U_1 U_2 F_1 dU_1 + U_1^2 F_1 dU_2 &= -d\lambda_2\ . \end{aligned} \tag{D.6}$$

The first consequence is that this system can be satisfied only if

$$U_1 d\lambda_1 + U_2 d\lambda_2 = 0\ , \tag{D.6a}$$

which shows that U_1 and U_2 are defined only as long as λ_1 and λ_2 satisfy a given relation. Equation D.6a, together with (D.5) gives $dH = 0$ and is consistent with our previous remark that H has to take a fixed value, namely zero.

We assume thus that λ_1 and λ_2 are varied simultaneously according to (D.6a). Then the two equations (D.6) are redundant. But we can arbitrarily decide to choose U_1 and U_2 such that

$$U_1^2 + U_2^2 = 1 \tag{D.7}$$

since their ratio, giving the direction (U_1, U_2), only matters. Then, we have

$$U_1 dU_1 + U_2 dU_2 = 0 \tag{D.7a}$$

and placing that in (D.6), we find, making use of (D.7)

$$\frac{\partial U_2}{\partial \lambda_2} = \frac{-1}{F_1} \tag{D.8}$$

where U_2 is now a function of x_1, x_2, and λ_2.

This relation could have been obtained starting from the beginning with

$$\begin{aligned} \dot{x}_1 &= \cos \gamma \\ \dot{x}_2 &= \sin \gamma \end{aligned} \tag{D.1a}$$

instead of (D.1), and having γ as only control. But this approach yields less information. It does not clearly show the existence of a contraint or the λ's, and the equality $H = 0$ must be borrowed from the general theory to establish (D.8).

We see that under condition (D.3), U_2 is continuous in λ_2. Moreover, from our assumptions on the regularity of F, under condition (D.7) F_1 is bounded from above. Therefore, in view of (D.5a), we

have, for some fixed m

$$\frac{\partial U_2}{\partial \lambda_2} = -\frac{1}{F_1} < m < 0 . \tag{D.9}$$

This still does not prove that $U_2 = 0$ has a solution, because the added condition on the λ's can restrict the possible values of λ_2 to a finite interval, or to a semi-infinite one, such as $\lambda_2 \leq \lambda_2^o$ (or $\lambda_2 \geq \lambda_2^o$).

Assume that U_2 is defined for some (λ_1, λ_2). Vary λ_2 by $d\lambda_2$, and we see that we can find the corresponding variation $d\lambda$, from (D.6a), except when $U_1 = 0$. In that case, from (D.7), $U_2^2 = 1$. Assume that the allowable λ_2's have an upper bound of λ_2^o. Then, as λ_2 increases toward λ_2^o, U_2 decreases toward $U_2^o = \pm 1$. And as u_2 cannot be larger than one, we necessarily have $u_2^o = -1$. Similarly, if λ_2 has a lower bound, it corresponds to $U_2 = +1$. This, together with the continuity of U_2 and with (D.9), shows that the equation $U_2 = 0$ always has a (single) root in λ_2. Therefore, our condition (D.4) is satisfied.

The conclusion is that Weierstrass' condition implies ours, which is thus more general. However, it must be noticed that the earlier is much more elegant than the latter in that it can be readily checked from the data of the problem, without actually determining $U_2(x_1,x_2,\lambda_2)$.

Similar simple conditions might exist for the control problem but we did not investigate this question. However, it is interesting to notice a typical case where our condition is not met: when a control determined by the state constraint otherwise has a "bang-bang" behavior (and except if the constraint happens to imply an extreme value of the control). A typical example is the Dolichobrachistochrone problem.

In these cases, the additional adjoint is actually provided by singular arc conditions. This shows the close relation between singular solutions and the question of how a state constraint is joined on.

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